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Holographic flavour in the $\mathcal{N} = 1$ Polchinski-Strassler background

Christoph Sieg[1]

Università degli Studi di Milano
Via Celoria 16, 20133 Milano, Italy

Abstract

To endow the $\mathcal{N} = 1^\star$ SYM theory with quarks, we embed D7-brane probes into its gravity dual, known as the Polchinski-Strassler background. The non-vanishing 3-form flux G_3 in the background is dual to mass terms for the three adjoint chiral superfields, deforming the $\mathcal{N} = 4$ SYM theory to the $\mathcal{N} = 1^\star$ SYM theory. We keep its three mass parameters independent. This generalizes our analysis in hep-th/0610276 for the $\mathcal{N} = 2^\star$ SYM theory. We work at second order in the mass perturbation, i.e. G_3 and its backreaction on the background are considered perturbatively up to this order. We find analytic solutions for the embeddings which in general depend also on angular variables. We discuss the properties of the solutions and give error estimates on our approximation. By applying the method of holographic renormalization, we show that in all cases the embeddings are at least consistent with supersymmetry.

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[1] csieg@mi.infn.it

Contents

1 Introduction

The AdS/CFT correspondence [1] allows us to study $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with gauge group $SU(N)$ in the large N limit and at large 't Hooft coupling constant $\lambda = g_{\mathrm{YM}}^2 N$ by analyzing its conjectured gravity dual, given by type II B supergravity in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with N units of Ramond-Ramond 5-form flux.

The correspondence has been extended to cases in which the dual gauge theory is not maximally supersymmetric and conformal, but preserves less supersymmetries and is confining. Several backgrounds for the dual gravity description of such gauge theories have been proposed e.g. in [2–5].

A particularly interesting example for a supergravity background was discussed by Polchinski and Strassler [6]. It is based on the observation that from the $\mathcal{N} = 4$ SYM theory one can obtain a confining gauge theory with less supersymmetry by adding mass terms for the three adjoint chiral $\mathcal{N} = 1$ supermultiplets. In the dual gravity description this mass perturbation corresponds to certain non-vanishing 3-form flux components. In the underlying brane picture this flux polarizes the background generating D3-branes due to the Myers effect [7] into their transverse directions in which they extend to rotational ellipsoids [8]. The full effective description of the brane configuration is not known. The Polchinski-Strassler background which should be obtained as its near horizon limit therefore is not given in a closed form. At sufficiently large distance from the extended D3-brane sources the near horizon limit of the configuration is given as perturbative expansion around $\mathrm{AdS}_5 \times \mathrm{S}^5$. The 3-form flux is considered as a perturbation. Its backreaction on the geometry corrects the background order by order in the mass parameters [6, 9, 10].

Generically the dual gauge theory is the so called $\mathcal{N} = 1^*$ theory. Its special case when all three masses are identical has been discussed by Polchinski and Strassler [6]. If instead two masses are identical and non-vanishing, while the third one is zero, the theory is the $\mathcal{N} = 2^*$ theory. By introducing a fourth mass for the gravitino into the $\mathcal{N} = 1^*$ theory, supersymmetry can be completely broken. This case has been addressed in [11].

All the fields in the above mentioned gauge theories transform in the adjoint representation of the gauge group. To approach a dual gravity description of QCD we should extend the field content by adding fields that transform in the fundamental representation (henceforth denoted as quarks). It was proposed by Karch and Katz [12] that $\mathcal{N} = 4$ SYM can be endowed with N_{f} quark flavours by embedding N_{f} spacetime-filling D7-branes into $\mathrm{AdS}_5 \times \mathrm{S}^5$. In the brane picture, the N_{f} quark flavours correspond to open strings that connect the stack of N D3-branes with the N_{f} D7-branes. Taking the near horizon limit to obtain the gravity background for the correspondence, the gauge

symmetry on the D7-branes becomes the global flavour symmetry. The choice $N_{\rm f} \ll N$ thereby allows one to neglect the backreaction of the D7-branes on the background, considering them as brane probes. Each D7-brane probe spans an $\mathrm{AdS}_5 \times \mathrm{S}^3$ inside $\mathrm{AdS}_5 \times \mathrm{S}^5$. It fills all of AdS_5 down to a minimal value $r = \hat{u}$ of the radial coordinate r, at which it terminates. Since r has the interpretation of an energy scale with small and large r corresponding to the IR and UV regimes in the dual gauge theory, the value $r = \hat{u}$ is related to the quark mass $m_{\rm q}$ via $m_{\rm q} = \frac{1}{2\pi\alpha'}\hat{u}$. The termination of the D7-brane at $r < u$ means that at energies $E < m_{\rm q}$ the corresponding quark degree of freedom freezes out. Furthermore, the fluctuations of the D7-brane embedding coordinates around the found solution determine the meson spectrum in the dual gauge theory [13].

In the context of the AdS/CFT correspondence the embeddings of Dp-brane probes into various supergravity backgrounds have been studied extensively in the literature [14–26]. Analyses beyond the probe approximation have also been performed [16,27–32]. Embeddings of Dp-branes in backgrounds with flux have been treated in [8, 33, 34].

To add flavour to the mass perturbed $\mathcal{N} = 1^\star$ and $\mathcal{N} = 2^\star$ theories, we study in this paper the embedding of D7-brane probes into the order $\mathcal{O}(m^2)$ Polchinski-Strassler background with generic mass parameters. This generalizes our analysis in [8] in which we restricted ourselves to two specific embeddings in the $\mathcal{N} = 2$ background. In the generic case the presence of anti-selfdual source terms in the equation of motion for F disposes us to revisit the solution and also the treatment of the gauge field in [8]. Before studying the embedding coordinates themselves, we introduce additional parameters into the underlying action which allow us to reproduce the results of [8] even after modifying the treatment of F. Furthermore, we can directly see how the individual contributions to the action influence the behaviour of the embeddings.

The paper is organized as follows. In section 2 we review in brief the Polchinski-Strassler background with arbitrary mass parameters up to order $\mathcal{O}(m^2)$. In section 3 we present the expanded generalized form of the D7-brane action on which the whole analysis is based. In section 4 we revisit the equation of motion for the the D7-brane worldvolume gauge field and its solution. In section 5 we evaluate the action for the expanded embeddings and discuss the resulting equations of motion and their general regular solutions. We also revisit the case of [8] with our new treatment of the gauge field and analyze additional embeddings in the $\mathcal{N} = 2$ and in the $\mathcal{N} = 1$ background with three equal masses in more detail. Moreover, we give error estimates for the analytic solutions. In section 6 we apply the method of holographic renormalization [35–37] to the on-shell action of arbitrary D7-brane embeddings in the generalized background and show that in all cases the subtracted action can be made vanish by adding appropriate finite counterterms. Various detailed computations that include the generalization of the background to arbitrary mass parameters, the derivation of the explicit form of the

equations of motion and of the action as well as the derivation of their solutions can be found in a series of appendices.

2 Polchinski-Strassler background to order $\mathcal{O}(m^2)$

In the following we work in the regime in which the Polchinski-Strassler background [6] can be described as a perturbative expansion around $\mathrm{AdS}_5 \times \mathrm{S}^5$. The corrections are determined by the backreaction of the 3-form flux on the geometry. In the Einstein frame of [6, 8] the unperturbed metric of $\mathrm{AdS}_5 \times \mathrm{S}^5$ reads

$$\mathrm{d}s^2 = Z^{-\frac{1}{2}} \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu + Z^{\frac{1}{2}} \delta_{ij} \, \mathrm{d}y^i \, \mathrm{d}y^j \ ,$$

$$Z(r) = \frac{R^4}{r^4} \ , \qquad r^2 = y^i y^i \ , \qquad R^4 = 4\pi g_\mathrm{s} N \alpha'^2 \ , \tag{2.1}$$

where $\mu, \nu = 0, 1, 2, 3$ and $i, j = 4, \ldots, 9$. The radius R depends on the string coupling constant g_s and on the number N of D3-branes which in the near horizon limit generate the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. The unperturbed background also contains the complex combination of the axion and dilaton and the 4-form potential which are defined as

$$\hat{\tau} = \hat{C}_0 + i\,\mathrm{e}^{-\hat{\phi}} = \mathrm{const.} \ , \qquad \hat{C}_{0123} = \mathrm{e}^{-\hat{\phi}} Z^{-1} \ , \tag{2.2}$$

where in the following a 'hat' always denotes an unperturbed quantity. The unperturbed dilaton is related to the string coupling constant as $\mathrm{e}^{\hat{\phi}} = g_\mathrm{s}$.

Polchinski and Strassler [6] have considered a perturbation in the form of a non-vanishing 3-form flux G_3 given by

$$G_3 = \tilde{F}_3 - \hat{\tau} H_3 = \mathrm{e}^{-\hat{\phi}} \frac{\zeta}{3} \, \mathrm{d}(Z S_2) \ , \tag{2.3}$$

where $\tilde{F}_3$ and H_3 are the 3-form field strengths, which are respectively obtained from the potentials

$$\tilde{C}_2 = C_2 - \hat{C}_0 B = \mathrm{e}^{-\hat{\phi}} \frac{\zeta}{3} Z \, \mathrm{Re}\, S_2 \ , \qquad B = -\frac{\zeta}{3} Z \, \mathrm{Im}\, S_2 \ . \tag{2.4}$$

The constant ζ assumes the value $\zeta = -3\sqrt{2}$ in a proper normalization scheme [6]. The 2-form S_2 has the component expression

$$S_2 = \frac{1}{2} T_{ijk} y^i \, \mathrm{d}y^j \wedge \mathrm{d}y^k \ , \tag{2.5}$$

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where the 3-tensor T_3 is imaginary anti-selfdual (IASD), i.e. it fulfills

$$(\star_6 + i)T_3 = 0 . \tag{2.6}$$

Thereby $\star_6$ is the Hodge star operator in flat space with components in the directions y^i of (2.1).

To present the explicit form of T_3 it is advantageous to work in a basis of three complex coordinates z^p and their complex conjugates $\bar{z}^p$ for the transverse directions y^i. It is defined as

$$z^p = \frac{1}{\sqrt{2}}(y^{p+3} + iy^{p+6}) , \qquad p = 1, 2, 3 . \tag{2.7}$$

The components of the tensor T_3 then read

$$T_{pqr} = T_{\bar{p}\bar{q}\bar{r}} = T_{\bar{p}qr} = 0 , \qquad T_{p\bar{q}\bar{r}} = \epsilon_{pqr}m_p , \tag{2.8}$$

where in the dual gauge theory the three parameters m_p are the masses of the three adjoint chiral $\mathcal{N} = 1$ multiplets Φ_p of $\mathcal{N} = 4$ SYM. To be more precise, the G_3 perturbation (2.3) is dual to a deformation of $\mathcal{N} = 4$ SYM by a mass-term superpotential

$$\Delta W = \frac{1}{g_{\mathrm{YM}}^2}(m_1 \operatorname{tr} \Phi_1^2 + m_2 \operatorname{tr} \Phi_2^2 + m_3 \operatorname{tr} \Phi_3^2) , \tag{2.9}$$

where $g_{\mathrm{YM}}^2 = 4\pi g_{\mathrm{s}}$. For generic masses the theory is $\mathcal{N} = 1$ supersymmetric, while for one mass vanishing and the other two being equal it preserves $\mathcal{N} = 2$ supersymmetries.

The mass perturbation (2.3) backreacts on the geometry. Up to linear order in the masses only 6-form potentials [6] are induced. At quadratic order in the masses, the metric and 4-form potential [9] as well as the complex axion-dilaton [6, 9, 10] acquire corrections.[2] Furthermore, a non-vanishing 8-form RR potential C_8 is induced [8]. The linear combination (2.3) still contains only the constant unperturbed $\hat{\tau}$, which changes at order $\mathcal{O}(m^3)$, at which a component $G_{(3,0)}$ is generated that is dual to a non-vanishing gaugino condensate [10].

The deformations at quadratic order for the metric, C_4 and τ have been computed in [9] with an appropriate gauge choice. At this order, the deformed metric reads

$$\mathrm{d}s^2 = (Z^{-\frac{1}{2}} + h_0)\eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu + \left[(5Z^{\frac{1}{2}} + p)I_{ij} + (Z^{\frac{1}{2}} + q)\frac{y^i y^j}{r^2} + wW_{ij}\right] \mathrm{d}y^i \, \mathrm{d}y^j , \tag{2.10}$$

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[2]See [8] for remarks about some typos in the original papers.

where the tensors I_{ij} and W_{ij} are given by

$$I_{ij} = \frac{1}{5}\left(\delta_{ij} - \frac{y^i y^j}{r^2}\right) , \qquad W_{ij} = \frac{1}{|T_3|^2}\operatorname{Re}(T_{ipk}\bar{T}_{jpl})\frac{y^k y^l}{r^2} - I_{ij} , \qquad |T_3|^2 = \frac{1}{3!}T_{ijk}\bar{T}_{ijk} .$$
$$(2.11)$$

It is important to remark that our definition of $|T_3|^2$ deviates from the one in [9] by an extra factor $\frac{1}{3!}$, such that we have the relation

$$|T_3|^2 = m_1^2 + m_2^2 + m_3^2 = M^2 . \tag{2.12}$$

The functions h, w, p, q are given by [9]

$$w = -\frac{\zeta^2 M^2 R^2}{18}Z , \qquad p = -\frac{\zeta^2 M^2 R^2}{48}Z , \qquad q = \frac{\zeta^2 M^2 R^2}{1296}Z , \qquad h_0 = \frac{7\zeta^2 M^2 R^2}{1296} ,$$
$$(2.13)$$

and they satisfy

$$4h_0 Z = q - p . \tag{2.14}$$

The correction to the dilaton $\tilde{\phi} = \varphi Y_+$ is given as a product of a purely radial dependent part φ and an $SO(6)$ spherical harmonic Y_+, which explicitly read

$$\varphi = \frac{\zeta^2 M^2 R^2}{108}Z^{\frac{1}{2}} , \qquad Y_+ = \frac{3}{M^2 r^2}\left(m_2 m_3(y_4^2 - y_7^2) + m_1 m_3(y_5^2 - y_8^2) + m_1 m_2(y_6^2 - y_9^2)\right) .$$
$$(2.15)$$

The (backreacted) forms to order $\mathcal{O}(m^2)$ relevant here are given by the 2-form potentials (2.4) and by

$$C_4 = e^{-\hat{\phi}}\left(Z^{-1} + \frac{\zeta^2 M^2 R^2}{3^4 2^3}Z^{-\frac{1}{2}}\right)\operatorname{dvol}(\mathbb{R}^{1,3}) + \frac{1}{2}B \wedge C_2 ,$$

$$C_6 = \frac{2}{3}B \wedge \hat{C}_4 , \tag{2.16}$$

$$C_8 = -\frac{1}{6}\left(e^{2\hat{\phi}}\tilde{C}_2 \wedge \tilde{C}_2 + B \wedge B\right) \wedge \hat{C}_4 ,$$

where again $\hat{C}_4$ denotes the unperturbed 4-form potential in (2.2) which is the first term in the expression for C_4 above. It turns out that only $\hat{C}_4$ of C_4 is relevant for a D7-brane embedding up to order $\mathcal{O}(m^2)$.

	$x^\mu, \mu = 0, 1, 2, 3$	$y^i, i = 4, 5, 6, 7, 8, 9$	
		$y^a, a = 5, 6, 8, 9$	$y^m, m = 4, 7$
D3	$-$	$\cdot$	$\cdot$
D7	$-$	$-$	$\cdot$

Table 1: Orientation of the D7-brane probe w.r.t. the background generating stack of D3-branes in absence of the mass perturbation. The D3-branes are then localized (denoted by '·') in the six transverse directions, while the D7-branes fill four of these directions (denoted by '$-$').

3   The action

The action for a D7-brane is given by the sum of the Dirac-Born-Infeld (DBI) and Chern-Simons (CS) action, i.e.

$$S = S_{\mathrm{DBI}} + S_{\mathrm{CS}} \ , \tag{3.1}$$

$$S_{\mathrm{DBI}} = -\frac{T_7}{\mathrm{e}^{2\hat\phi}} \int \mathrm{d}^8\xi \, \mathrm{e}^\phi \sqrt{\left| \det\left(P[g] + 2\pi\alpha' \, \mathrm{e}^{-\frac{\phi-\hat\phi}{2}} \, \mathcal{F}\right)\right|} \ , \tag{3.2}$$

$$S_{\mathrm{CS}} = -\mu_7 \int \sum_{r=1}^{4} P[C_{2r}] \wedge \mathrm{e}^{2\pi\alpha'\mathcal{F}} \ , \tag{3.3}$$

where $T_7 = \mu_7$ and the expressions are given in the Einstein frame which is related to the string frame by using only the non-constant part $\tilde\phi = \phi - \hat\phi$ of the dilaton ϕ. The field strength $\mathcal{F}$ is a linear combination of the field strength $F = \mathrm{d}A$ of the worldvolume gauge potential A and the pullback of B as

$$2\pi\alpha'\mathcal{F} = 2\pi\alpha' F - P[B] \ . \tag{3.4}$$

In [8] we have introduced the minus sign in (3.3) for physical reasons. We also assumed there that it should be the right choice to preserve some supersymmetries of the background. Here our modified treatment of the gauge field in general alters the embeddings. However, based on the unaffected y^4 embeddings, we can still favour this sign choice. At some points we nevertheless also discuss the effects of the alternative choice. For a final decision, a check of the κ-symmetry on the worldvolume of the D7-brane is required, which we leave as an open problem.

In the coordinate system used in (2.1) the background generating stack of D3-branes and the D7-brane probe are oriented as shown in figure 1. For embedding coordinates

that do not depend on the four worldvolume directions x^μ, the pullbacks are non-trivial only in the additional four directions labeled by y^a. In static gauge, the pullback of a generic 2-tensor E_{ij} on these directions reads

$$P[E]_{ab} = E_{ab} + \partial_a X^m E_{mb} + \partial_b X^n E_{an} + \partial_a X^m \partial_b X^n E_{mn} \ . \tag{3.5}$$

Expanding the complete D7-brane action to quadratic order in the mass perturbation around the unperturbed background (2.1) and (2.2), we find

$$S = -\frac{T_7}{\mathrm{e}^{\hat{\phi}}} \int \mathrm{d}^8\xi \sqrt{\det P[\delta]} \Big[1 + \tilde{\phi} + \frac{1}{2} Z^{\frac{1}{2}} \tilde{g}_{\mu\mu} + \frac{1}{2} Z^{-\frac{1}{2}} P[\delta]^{ab} P[\tilde{g}]_{ab}$$

$$+ \frac{1}{2} Z^{-1} \big((\alpha - \beta\star_4) P[B] \cdot P[B] - 4\pi\alpha'(\mu - \nu\star_4) F \cdot P[B]$$

$$+ 4\pi^2 \alpha'^2 (1 + \star_4) F \cdot F + \tau\, \mathrm{e}^{2\hat{\phi}} P[\tilde{C}_2] \cdot \star_4 P[\tilde{C}_2] \big) \Big] \ , \tag{3.6}$$

where we have introduced constants which in the case of the Polchinski-Strassler background take values

$$\alpha = 1 \ , \qquad \beta = \frac{2}{3} \ , \qquad \mu = 1 \ , \qquad \nu = -\frac{1}{3} \ , \qquad \tau = -\frac{1}{3} \ , \tag{3.7}$$

where τ must not be confused with the complex axion-dilation defined in (2.2). Furthermore, throughout the paper with a 'tilde' we denote the order $\mathcal{O}(m^2)$ corrections[3] to the unperturbed quantities which carry a 'hat'. We should stress that here the four-dimensional inner product $\cdot$ as well as the Hodge star $\star_4$ in (3.6) are understood to be computed with the pullback of the Kronecker delta denoted by $P[\delta]_{ab}$. For two generic 2-forms ω_2 and ω_2' they are defined as

$$\omega_2 \cdot \omega_2' = \frac{1}{2} P[\delta]^{a_1 b_1} P[\delta]^{a_2 b_2} \omega_{a_1 a_2} \omega_{b_1 b_2}' \ , \qquad \star_4 \omega_{a_1 a_2} = \frac{1}{2} \sqrt{\det P[\delta]} \epsilon_{a_1 a_2}{}^{b_1 b_2} \omega_{b_1 b_2} \ , \tag{3.8}$$

where $\epsilon_{5689} = 1$, and indices are raised with the inverse of $P[\delta]_{ab}$ denoted by $P[\delta]^{ab}$.

We have introduced the constants α, β, μ, ν, τ in (3.6) for two reasons. First of all, we want to keep the option to alter the corresponding values. This turns out to be necessary after the original treatment of the gauge field [8] has been modified as described in section 4. Secondly, keeping these constants makes it easy to identify how

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[3]By notational abuse, this does not apply to $\tilde{C}_2$ and its field strengths $\tilde{F}_3$.

the individual parts in the action contribute to the equations of motion and thus how they take influence on the embeddings.

With the above values of the parameters it is obvious that the action (3.6) does not longer depend on the gauge invariant combination (3.4) of the gauge field strength F and of the pullback of B. The additional dependence on B arises because the explicit expressions for C_6 and C_8 in (2.16) explicitly contain B. They are obtained if we make use of (2.4) which relates the 2-form potentials $\tilde{C}_2$ and B to S_2 an hence fixes their gauge freedom. This allows e.g. $P[B]$ to appear explicitly outside the combination $\mathcal{F}$ defined in (3.4). It also implies that the equations of motion for F do not contain F and $P[B]$ only as the combination $\mathcal{F}$.

4 The gauge field equation revisited

The equations of motion for the D7-brane embedding coordinates depend on F. To determine the embeddings, we therefore have to discuss also the equation of motion for F and its solution. As we have already shown in [8], it in general contains source terms which come from the terms linear in F in the action (3.6). In this section we will extend the discussion from the $\mathcal{N} = 2$ case to the generic $\mathcal{N} = 1$ case. With an expansion of the embedding coordinates we will show that up to order $\mathcal{O}(m^2)$ no source terms for F are present. For the $\mathcal{O}(m^2)$ result to suffice, we will have to modify our previous understanding [8] of the role of F.

A variation of the action (3.6) w.r.t. the gauge potential A gives the equation of motion

$$\mathrm{d}\big(\hat{C}_4 \wedge (2\pi\alpha'(\star_4 + 1)F - (\mu \star_4 - \nu)P[B])\big) = 0 \; , \tag{4.1}$$

where we have transformed inner products multiplied by the volume element into wedge products by using the Hodge star $\star_4$ and also the explicit unperturbed metric (2.1). Integrating the above expression and inserting the explicit expression for $\hat{C}_4$ in (2.2), we find

$$Z^{-1}(2\pi\alpha'(\star_4 + 1)F - (\mu \star_4 - \nu)P[B]) = \mathrm{d}P[\omega_1] \; . \tag{4.2}$$

We have introduced ω_1 to take into consideration the freedom in integrating the exterior derivative. By acting with the linear combination $1 \pm \star_4$, the above equation is separated into two equations according to

$$4\pi\alpha' Z^{-1}F_+ - (\mu - \nu)Z^{-1}P[B]_+ = \mathrm{d}P[\omega_1]_+ \; , \qquad (\mu + \nu)Z^{-1}P[B]_- = \mathrm{d}P[\omega_1]_- \; . \tag{4.3}$$

We have thereby used that the decomposition of a 2-form ω_2 in its selfdual and anti-

selfdual components ω_{2+} and respectively ω_{2-} is given by

$$\omega_2 = \omega_{2+} + \omega_{2-} \ , \qquad \omega_{2+} = \frac{1}{2}(1 + \star_4)\omega_2 \ , \qquad \omega_{2-} = \frac{1}{2}(1 - \star_4)\omega_2 \ . \qquad (4.4)$$

While the first equation in (4.3) contains F_+, the second one does not contain any degrees of freedom of F. For $\mu + \nu \neq 0$ this equation in general is a non-trivial constraint for the embedding coordinates, which enter via the pullback. It means that the pullback of $Z^{-1}B$ has to follow as exterior derivative of a 1-form. One therefore has to solve a coupled system of differential equations that consists of the equations of motion for the embedding coordinates and the two equations in (4.3). In the following we describe a solution which is based on perturbation theory.

We assume that, as the background itself, also the D7-brane embedding coordinates $y^m(y^a)$ can be treated perturbatively. The leading contributions are constants $\hat{y}^m$ that describe the constant embedding of D7-branes in pure $\mathrm{AdS}_5 \times \mathrm{S}^5$ found in [12]. In the Polchinski-Strassler background the embeddings are corrected at higher orders by non-constant contributions $\tilde{y}^m(y^a)$, such that we write

$$y^m(y^a) = \hat{y}^m + \tilde{y}^m(y^a) \ . \qquad (4.5)$$

In case of the $\mathcal{N} = 2$ background [8] the decomposition was used to expand the action and equations of motion for the embedding coordinates themselves. However, unlike here, it was not used for determining the gauge field.

The correction $\tilde{y}^m$ is of order $\mathcal{O}(m^2)$. Since B itself is of order $\mathcal{O}(m)$, the derivative terms in the pullbacks (3.5) in static gauge are therefore beyond the order $\mathcal{O}(m^2)$ up to which we consider the background and the equations of motion for y^m. The same holds for the pullback of the unperturbed diagonal metric (2.1), for which the terms linear in the derivatives vanish exactly. Thus, the Hodge star as defined in (3.8) reduces to the one in flat space. It is again advantageous to work in the complex basis (2.7), in which the D7-brane embeddings oriented as in table 1 are along z^a, $\bar{z}^a$, $a = 2, 3$ and the transverse embedding coordinates are given by z^m, $\bar{z}^m$, $m = 1$. In this basis, the imaginary selfdual and anti-selfdual components of any 2-form ω decompose as

$$\omega_{2+} = \omega_{(1,1)}^{\mathrm{P}} \ , \qquad \omega_{2-} = \omega_{(2,0)} + \omega_{(0,2)} + \frac{1}{2}\omega_{a\bar{a}} \, \mathrm{d}z^b \wedge \mathrm{d}\bar{z}^b \ , \qquad (4.6)$$

where P denotes the primitive part of ω_2, i.e. $\omega_{a\bar{a}}^{\mathrm{P}} = 0$, and summations over a and b are understood. The potential B is primitive. We therefore assume that so are F and $\mathrm{d}\omega_1$. The equations (4.3) then reduce to

$$4\pi\alpha' Z^{-1}F_{(1,1)} - (\mu - \nu)Z^{-1}B_{(1,1)}^{\|} = \mathrm{d}\omega_{(1,1)}^{\|} \ , \qquad (\mu + \nu)Z^{-1}B_{(2,0)}^{\|} = \mathrm{d}\omega_{(2,0)}^{\|} \ , \qquad (4.7)$$

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where by $\|$ we denote the components of the corresponding form which are parallel to the directions of the D7-brane. Up to order $\mathcal{O}(m^2)$ we write

$$\operatorname{Im} S^{\|}_{(1,1)} = -\frac{i}{2}(T_{\bar{m}a\bar{b}}\hat{\bar{z}}^m - \bar{T}_{m\bar{a}b}\hat{z}^m)\,\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^b\ , \qquad \operatorname{Im} S^{\|}_{(2,0)} = -\frac{i}{4}\bar{T}_{\bar{m}ab}\hat{\bar{z}}^m\,\mathrm{d}z^a \wedge \mathrm{d}z^b\ , \tag{4.8}$$

which according to (2.4) up to a constant factor are the components of $Z^{-1}B^{\|}$. The two expressions follow as exterior holomorphic or anti-holomorphic derivatives of 1-form potentials. With $\mathrm{d} = \partial + \bar{\partial}$, we hence find that the choice

$$\omega^{\|}_1 = -i\frac{\zeta}{12}\Big(2(\mu - \nu)(T_{\bar{m}\bar{a}b}\bar{z}^a\hat{\bar{z}}^m\,\mathrm{d}z^b - \bar{T}_{m\bar{a}b}z^a\hat{z}^m\,\mathrm{d}\bar{z}^b) \\ - (\mu + \nu)(\bar{T}_{\bar{m}ab}z^a\hat{\bar{z}}^m\,\mathrm{d}z^b - T_{m\bar{a}\bar{b}}\bar{z}^a\hat{z}^m\,\mathrm{d}\bar{z}^b)\Big) \tag{4.9}$$

allows us to gauge away all source terms for F such that to order $\mathcal{O}(m^2)$ the equations (4.7) are consistently solved if F obeys

$$\mathrm{d}(Z^{-1}F_+) = 0\ , \qquad \mathrm{d}F = 0\ , \tag{4.10}$$

where the second relation is the Bianchi identity. Clearly, both equations are compatible with a vanishing gauge field $F = 0$ on the D7-brane, which is what we will assume from now on.

Up to order $\mathcal{O}(m^2)$ also the solution for F found in [8] in the case of the $\mathcal{N} = 2$ background solves the above equation. Moreover, it contains also $\mathcal{O}(m^3)$ terms, which we had to keep since we did not consider F as an independent field. This means, we have plugged the found F into the action before deriving the equations of motion for the embedding coordinates. The $\mathcal{O}(m^3)$ terms of F that contained derivatives of the embedding coordinates then contributed to the order $\mathcal{O}(m^2)$ embedding equations of motion. However, we should have better regarded F as an independent field and hence have inserted the result for F into the equations of motion for y^m. This procedure requires the result for F up to order $\mathcal{O}(m^2)$ only.

5 The expanded embeddings

5.1 Expanded action, equations of motion and solutions

In section 4 we have already made use of the expansion of the embedding into the constant unperturbed part $\hat{y}^m$ and the order $\mathcal{O}(m^2)$ correction (4.5). Inserting this

decomposition into (3.6), the pullbacks of the Kronecker δ simplify to the Kronecker δ on the worldvolume of the D7-brane. Since the equations of motion are found by taking derivatives w.r.t. $\tilde{y}^m$ and $\partial_a \tilde{y}^m$, one has to keep those terms which contribute up to order $\mathcal{O}(m^2)$ to the equations, even if they are of higher order in the action. The action (3.6) is then expanded as

$$
\begin{aligned}
S = -\frac{T_7}{\mathrm{e}^{\hat{\phi}}} \int \mathrm{d}^8\xi \Big[1 + \tilde{\phi} + \frac{1}{2} Z^{\frac{1}{2}} \tilde{g}_{\mu\mu} + \frac{1}{2} Z^{-\frac{1}{2}} \tilde{g}_{aa} + \frac{1}{2}(\partial_a \tilde{y}^m)^2 + Z^{-\frac{1}{2}} \partial_a \tilde{y}^m \tilde{g}_{ma} \\
+ \frac{1}{2} Z^{-1}\big((\alpha - \beta\star_4) B \cdot B + 4(\gamma - \delta\star_4) B \cdot \partial\tilde{y}B \\
- 4\pi\alpha'(\mu - \nu\star_4) F \cdot (B + 4\partial\tilde{y}B) + 4\pi^2\alpha'^2(1 + \star_4) F \cdot F \\
+ \tau\, \mathrm{e}^{2\hat{\phi}} \star_4 \tilde{C} \cdot (\tilde{C} + 4\partial\tilde{y}\tilde{C}) \big) \Big] \, ,
\end{aligned}
$$

$$(5.1)$$

where the inner product and the Hodge star operator are computed w.r.t. the flat 4-dimensional metric. We have furthermore used the abbreviations

$$
(\partial\tilde{y}B)_{ab} = \partial_a \tilde{y}^m B_{mb} \, , \qquad (\partial\tilde{y}\tilde{C})_{ab} = \partial_a \tilde{y}^m \tilde{C}_{mb} \, , \tag{5.2}
$$

and in addition we have introduced the constants

$$
\gamma = 1 \, , \qquad \delta = \frac{2}{3} \, , \tag{5.3}
$$

which in our case take the same values as respectively α and β in (3.7). As already explained at the end of section 4, inserting the non-vanishing solution for the gauge field found in [8] directly into the action alters some terms that contain derivatives of the embedding coordinates. The values of two parameters γ and δ then become

$$
\gamma = \frac{1}{3} \, , \qquad \delta = \frac{4}{3} \, , \tag{5.4}
$$

while the F-dependent terms then have to be removed from (5.1). To describe both cases, we keep γ and δ as independent constants.

The equations of motions for the embedding coordinates, which follow from the

action (5.1), are given by

$$\partial_a\left(\partial_a\tilde{y}^m + Z^{-\frac{1}{2}}\tilde{g}_{ma} + Z^{-1}\big((\gamma - \delta\star_4)B_{ab}B_{mb} - 2\pi\alpha'(\mu - \nu\star_4)F_{ab}B_{mb} + \tau\,\mathrm{e}^{2\hat{\phi}}\star_4\tilde{C}_{ab}\tilde{C}_{mb}\big)\right)$$

$$= \frac{\partial}{\partial\tilde{y}^m}\left(\tilde{\phi} + \frac{1}{2}Z^{\frac{1}{2}}\tilde{g}_{\mu\mu} + \frac{1}{2}Z^{-\frac{1}{2}}\tilde{g}_{aa} + \frac{1}{2}Z^{-1}\big((\alpha - \beta\star_4)B\cdot B + \tau\,\mathrm{e}^{2\hat{\phi}}\star_4\tilde{C}_2\cdot\tilde{C}_2\big)\right)\Big|_{\tilde{y}^m=0}$$

$$+ 2\pi\alpha'\left(\pi\alpha'\frac{\partial}{\partial\tilde{y}^m}Z^{-1}(1+\star_4)F\cdot F - (\mu-\nu\star_4)F\cdot\frac{\partial}{\partial\tilde{y}^m}(Z^{-1}B)\right)\Big|_{\tilde{y}^m=0}.$$

$$(5.5)$$

The solution $F = 0$ of (4.1) is from now on inserted into the above equations.

It is advantageous to introduce polar coordinate systems for the four worldvolume directions y^a of the D7-brane and for the two transverse directions y^m. The radial coordinate r of the full six-dimensional transverse space as defined in (2.1) splits into the radii ρ on the D7-brane worldvolume and u of the two transverse embedding directions according to

$$r = \sqrt{\rho^2 + u^2}\,, \qquad \rho = \sqrt{y^a y^a}\,, \qquad u = \sqrt{y^m y^m}\,, \qquad (5.6)$$

where summations over a and m are understood. In the polar coordinate system with angular coordinate ψ the two embedding coordinates y^m read

$$y^4 = u\cos\psi = \hat{u}\cos\hat{\psi} - \hat{u}\tilde{\psi}\sin\hat{\psi} + \tilde{u}\cos\hat{\psi}\,,$$
$$y^7 = u\sin\psi = \hat{u}\sin\hat{\psi} + \hat{u}\tilde{\psi}\cos\hat{\psi} + \tilde{u}\sin\hat{\psi}\,. \qquad (5.7)$$

In the final equalities we have expanded up to linear order in the corrections $\tilde{u}$ and $\tilde{\psi}$ to the unperturbed radius $\hat{u}$ and angle $\hat{\psi}$ which also present the boundary values at $\rho \to \infty$ of the embedding functions. In the dual gauge theory $\hat{u}$ determines the mass m_{q} of the quarks via $m_{\mathrm{q}} = \frac{1}{2\pi\alpha'}\hat{u}$.

In appendix B we derive the equations of motion (5.5) in the above coordinate system. They assume the same form for the radial coordinate u as also for the angle ψ, such that we can compactly write

$$2\partial_a\partial_{\bar{a}}f = \frac{n_f}{\hat{r}^4}\left(B_f + C_f\frac{\hat{u}^2}{\hat{r}^2} - C_f^I y_I\frac{\rho^2}{\hat{r}^2}\right)\,, \qquad (5.8)$$

where we set either $f = u$ or $f = \psi$ and identify the normalization factor with $n_u = \hat{u}$ and $n_\psi = 1$, respectively. The r.h.s. depends on ρ explicitly and implicitly via the total radius $\hat{r}$ which is found from (5.6) when only the unperturbed parts $\hat{y}^m$ of the embedding coordinates are inserted. The dependence on the three angles in the four worldvolume directions y^a of the D7-brane is encoded within four of the nine $l = 2\ SO(4)$ spherical harmonics y_I which are defined in (B.16). The constants B_f, C_f and C_f^I which depend

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on the masses, the parameters (3.7) and (5.3) and the angle $\hat{\psi}$ are given in (B.21) for $f = u$ and in (B.22) for $f = \psi$.

In appendix D we show that, fixing the boundary value to $\hat{f}$, the above differential equation admits a unique analytic regular solution. Together with its derivatives it is given in (D.10) and reads

$$f = \hat{f} - \frac{n_f}{8}\left(B_f\frac{2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + C_f\frac{1}{\hat{r}^2} - C_f^I\left(\frac{2}{\rho^2}\left(1 - \frac{\hat{u}^2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2}\right) - \frac{1}{\hat{r}^2}\right)y_I\right)$$

$$\partial_\rho f = \frac{n_f}{4}\left(B_f\frac{2}{\rho}\left(\frac{1}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} - \frac{1}{\hat{r}^2}\right) + C_f\frac{\rho}{\hat{r}^4} - C_f^I\left(\frac{2}{\rho^3}\left(1 - 2\frac{\hat{u}^2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + \frac{\hat{u}^2}{\hat{r}^2}\right) - \frac{\rho}{\hat{r}^4}\right)y_I\right)$$

$$\partial_\rho^2 f = \frac{n_f}{4}\left(B_f\frac{2}{\rho^2}\left(-\frac{3}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + \frac{3}{\hat{r}^2} + 2\frac{\rho^2}{\hat{r}^4}\right) + C_f\frac{1}{\hat{r}^4}\left(1 - 4\frac{\rho^2}{r^2}\right)\right.$$
$$\left. - C_f^I\left(-\frac{20}{\rho^4}\left(1 - \frac{\hat{u}^2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2}\right) + \frac{1}{\rho^2\hat{r}^2}\left(10 + 3\frac{\rho^2}{\hat{r}^2}\right) + 4\frac{\rho^2}{\hat{r}^6}\right)\right)y_I\right) ,$$

(5.9)

where I is a summation index that runs over the four combinations that label the spherical harmonics y_I. The solution and their derivatives have asymptotic behaviours found in (D.11). The results read

$$f = \begin{cases} \hat{f} - \frac{n_f}{8\hat{u}^2}(2B_f + C_f) & \rho \to 0 \\ \hat{f} - \frac{n_f}{8\rho^2}\left(2B_f\ln\frac{\rho^2}{\hat{u}^2} + C_f - C_f^I y_I\right) & \rho \to \infty \end{cases} ,$$

$$\partial_\rho f = \begin{cases} 0 & \rho \to 0 \\ \frac{n_f}{4\rho^3}\left(2B_f(\ln\frac{\rho^2}{\hat{u}^2} - 1) + C_f - C_f^I y_I\right) & \rho \to \infty \end{cases} ,$$

(5.10)

$$\partial_\rho^2 f = \begin{cases} \frac{n_f}{12\hat{u}^2}(3(B_f + C_f) + C_f^I y_I) & \rho \to 0 \\ \frac{n_f}{4\rho^4}(2B_f(-3\ln\frac{\rho^2}{\hat{u}^2} + 5) - 3C_f + 3C_f^I y_I) & \rho \to \infty \end{cases} ,$$

where in case of the $\rho \to \infty$ limit we also have kept the next subleading contributions. Based on the above results the monotony properties of the solutions and also estimates of the deviation from the full numerical results are discussed in the following.

5.2   Monotony properties of the solutions

A physical embedding should lead to a monotonically increasing function $r(\rho)$ [24]. Taking the derivative of the total radius r as defined in (5.6) w.r.t ρ, and expanding the result up to order $\mathcal{O}(m^2)$, we find the condition

$$\hat{r}\partial_\rho r = \rho - \frac{\rho\hat{u}}{\hat{r}^2}\tilde{u} + \hat{u}\partial_\rho\tilde{u} \geq 0 .$$
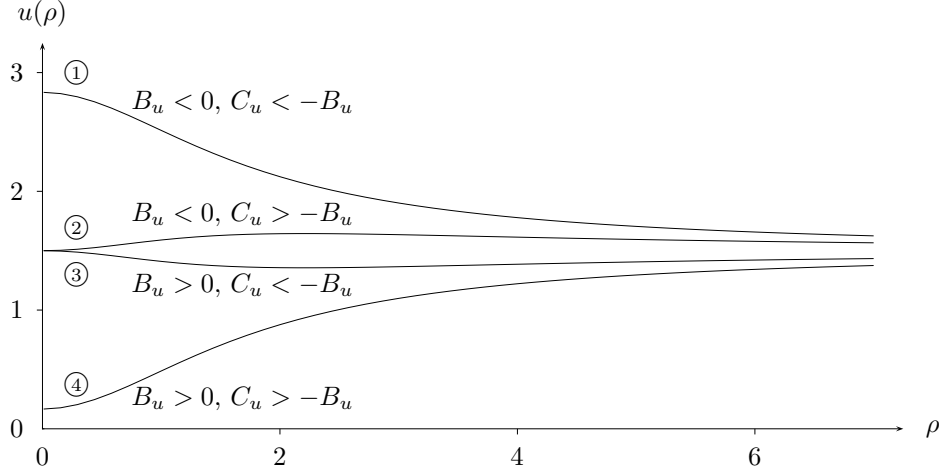
(5.11)

Figure 1: The four possible types of embeddings, presented for illustration with the values $\hat{u} = 1.5$, $mR^2 = 1$, $|C_u| = 2|B_u| = 8$.

Inserting the explicit result for $u(\rho)$ taken from (5.9), the result reads

$$\hat{r}\partial_\rho r = \rho + \frac{\hat{u}^2}{8}\bigg(B_u\frac{2}{\rho}\bigg(\bigg(\frac{1}{\hat{r}^2} + \frac{2}{\rho^2}\bigg)\ln\frac{\hat{r}^2}{\hat{u}^2} - \frac{2}{\hat{r}^2}\bigg) + 3C_u\frac{\rho}{\hat{r}^4}$$
$$+ C_u^I\bigg(3\frac{\rho}{\hat{r}^4} + 2\frac{\hat{u}^2}{\rho^3}\bigg(\frac{1}{\hat{r}^2} + \frac{4}{\rho^2}\bigg)\ln\frac{\hat{r}^2}{\hat{u}^2} + \frac{2}{\rho\hat{r}^2} - \frac{8}{\rho^3}\bigg)y_I\bigg) \; . \tag{5.12}$$

At large ρ, this expression is dominated by the first term, and therefore $r(\rho)$ is linearly increasing with ρ there. For $\rho \ll \hat{u}$ the above result expands as

$$\hat{r}\partial_\rho r = \bigg(1 + \frac{1}{24\hat{u}^2}(12B_u + 9C_u + 2C_u^I y_I)\bigg)\rho \; . \tag{5.13}$$

Since the constants B_u, C_u and C_u^I are proportional to m^2R^4, the derivative $\partial_\rho r$ becomes negative only if their linear combination is negative and if $\hat{u} \lesssim mR^2$ is sufficiently small to compensate the leading term. We do not investigate this further, since in the regime $\hat{u} \lesssim mR^2$ the analytic solution based on the expansion (4.5) cannot be trusted anyway, and therefore the exact result is required for a precise statement on the monotony properties. In any case, for $\hat{u}$ sufficiently large, one finds that $r(\rho)$ is monotonically increasing and hence the embedding is physical.

The requirement that $r(\rho)$ has to be a monotonically increasing function does not imply that $u(\rho)$ has to be monotonic, and in fact in general it is not. As shown in figure

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1, with $C_u^I = 0$ we find four distinct behaviours, depending on the relations between B_u and C_u. Two of them are monotonically increasing and respectively decreasing, while the other two assume an intermediate relative maximum or minimum. This is also visible from the asymptotic behaviour in (5.10). The transgression between a monotonic and non-monotonic embedding takes place at

$$C_u = -B_u \ . \tag{5.14}$$

It is interesting to analyze the behaviour of $u(\rho)$ under a relative sign flip between the DBI and CS action (3.2) and (3.3). We just have to compare the relations between B_u and C_u for both choices of the relative sign. In the combined expanded action (5.1) a change of the relative sign inverts the signs in front of all Hodge stars, i.e. the parameters β, δ and τ change their signs. As seen from (B.21), the coefficient C_u only depends on the combinations $\alpha \pm \beta - (\gamma \pm \delta)$. It is therefore insensitive to such a sign flip as long as $\beta = \delta$. The coefficient B_u differs from $B_u^{(-)}$, which is the one found in case of a relative minus sign between the DBI and CS action, as

$$B_u - B_u^{(-)} = \frac{\zeta^2 R^4}{18} \big((m_2^2 + m_3^2 - m_1^2)(\tau - \beta) + 2m_2 m_3 (\beta + \tau) \cos 2\hat\psi \big) \ , \tag{5.15}$$

where in the Polchinski-Strassler background β and τ assume the values given in (3.7). In particular, one has $\beta = -2\tau > 0$. If $m_2 = m_3 = m$ and either $m_1 = 0$ or $m_1 = m$ we find the inequality $B_u < B_u^{(-)}$ for arbitrary values of the angle $\hat\psi$. Therefore, a relative minus sign between the DBI and CS action leads to D7-brane embeddings which are more attracted towards the center of the space. If $m_1 = m_2 = m$ and $m_3 = 0$, B_u is independent of the relative sign choice, i.e. $B_u = B_u^{(-)}$.

In the following we keep the sign choice as in [8]. The y^4 embedding in the case $m_1 = 0$ and $m_2 = m_3 = m$ then is monotonically decreasing as a function of ρ. The alternative sign choice would alter this behaviour and lead to an intermediate minimum, as in the third case presented in figure 1. A numerical study reveals that the allowed radial boundary values $\hat u$ for which the y^4 embeddings obey $u(\rho) > 0$, differ for both choices of the sign. For the sign choice as in [8] the y^4 embeddings can assume all values $\hat u \geq 0$, while for the alternative sign choice $\hat u$ is restricted from below by $\hat u \geq \hat u_0 > 0$. There appears thus a gap in the allowed values for $\hat u$, separating the case $\hat u = 0$ from the continuum $\hat u \geq \hat u_0 > 0$. This is a disfavoured behaviour. We stress that the embeddings with $\hat u \simeq \hat u_0$ enter the region in which the expansion of the background itself breaks down. One must therefore not use this observation to completely rule out the possibility of a relative minus sign between the DBI and CS action. For a confirmed answer which of the sign choices is the correct one to preserve some supersymmetry, one has to check the kappa symmetry up to order $\mathcal{O}(m^2)$.

Inserting the explicit values $\alpha = \gamma = 1$, $\beta = \delta = \frac{2}{3}$ $\tau = -\frac{1}{3}$ for the Polchinski-Strassler background, we find that the only non-vanishing coefficients are given by

$$
\begin{aligned}
B_u &= -\frac{\zeta^2 R^4}{216}\left(3(m_2 + m_3)^2 + 5(m_2 - m_3)^2 - 4m_1^2 + 4m_2 m_3(1 - \cos 2\hat{\psi})\right) , \\
C_u &= -\frac{\zeta^2 R^4}{324}\left(7(m_2 + m_3)^2 - 11(m_2 - m_3)^2 - 4m_1^2 - 36m_2 m_3(1 - \cos 2\hat{\psi})\right) ,
\end{aligned}
\tag{5.16}
$$

or respectively

$$
\begin{aligned}
B_\psi &= -\frac{\zeta^2 R^4}{54} m_2 m_3 \sin 2\hat{\psi} , & C_\psi^{+-} &= \frac{\zeta^2 R^4}{54} m_1(m_2 + m_3) , \\
C_\psi &= \frac{2\zeta^2 R^4}{27} m_2 m_3 \sin 2\hat{\psi} , & C_\psi^{--} &= -\frac{\zeta^2 R^4}{54} m_1(m_2 - m_3) .
\end{aligned}
\tag{5.17}
$$

It is interesting to notice that the identification $\gamma = 1$ implies that $C_u^I = 0$ and $C_\psi^{\pm+} = 0$. This ensures that the radial embedding coordinate u remains independent of the angles in the D7-brane worldvolume coordinate system, regardless of the values of the masses and the other parameters. Furthermore, the dependence of the angular embedding coordinate ψ on the worldvolume angles is also reduced to only two spherical harmonics $y_{\pm -}$ in the generic mass case, and their corresponding coefficients $C_\psi^{\pm -}$ become independent of the unperturbed angle $\hat{\psi}$. For $m_2 = m_3$ the embedding depends only on y_{+-}, and for $m_1 = 0$ in any case ψ does not depend on any of the $SO(4)$ spherical harmonics.

5.3 The $\mathcal{N} = 2$ case with $m_1 = 0$ and $m_2 = m_3 = m$ revisited

If the mass parameters are given by $m_1 = 0$ and $m_2 = m_3 = m$, the dual gauge theory preserves $\mathcal{N} = 2$ supersymmetries. Adding D7-brane probes as indicated in table 1 should not break these supersymmetries. We denote them as $\mathcal{N} = 2_\parallel$ embeddings. Already at the end of section 4 we have stressed that in [8] the $\mathcal{N} = 2_\parallel$ embeddings have been studied by inserting the solution for the gauge field into the action before extracting the equations of motion for the embedding coordinates. The respective action is given by (5.1) with $F = 0$ and γ and δ assuming the values given in (5.4). However, we should consider the gauge field as an independent field and thus insert the solution for its field strength F into the equations of motion for the embedding coordinates. In this case we identify $\gamma = \alpha$ and $\delta = \beta$ with the explicit values given in (3.7). According to (B.21) and (B.22) this alters the values of C_u and C_ψ w.r.t. the ones in [8], while B_u and B_ψ
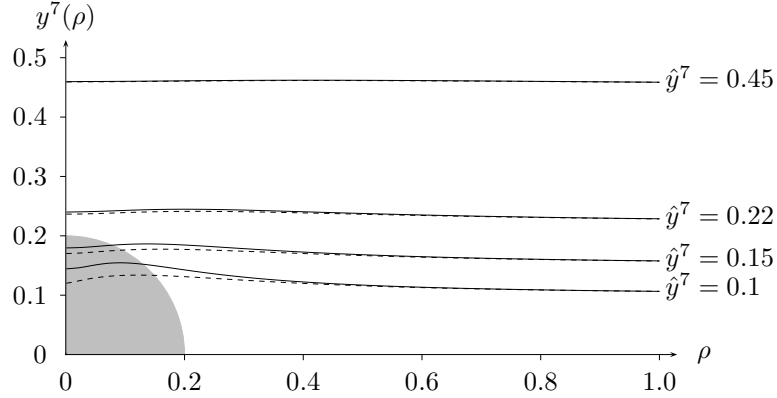
Figure 2: Analytic (solid line) and exact numerical (dashed line) embeddings along y^7 for $F = 0$ in the $\mathcal{N} = 2$ background with masses $m_1 = 0$, $m_2 = m_3 = m = 0.2$ and distinct boundary values $\hat{y}^7$. The grey quarter circle corresponds to $r \leq mR^2$, into which the background generating D3-branes are expected to expand. Lengths and masses are dimensionless and measured in units of R and R^{-1} respectively. The dimensionless boundary value $\hat{y}^7$ determines the dimensionful quark mass m_{q} according to $m_{\mathrm{q}} = \frac{R}{2\pi\alpha'}\hat{y}^7$.

are independent of γ and δ and thus remain unchanged. All the other coefficients vanish for $m_1 = 0$ anyway. The expressions which substitute the ones in [8] then read[4]

$$C_u = -\frac{\zeta^2 m^2 R^4}{81}(-2 + 9\cos 2\hat{\psi}) , \qquad C_\psi = \frac{2\zeta^2 m^2 R^4}{27}\sin 2\hat{\psi} . \qquad (5.18)$$

Our analytic solutions with $\hat{\psi} = 0$ here and in [8] are based on $F = 0$. As is seen directly from (B.21), with $\hat{\psi} = 0$ also C_u does not depend on γ and δ and $C_\psi = 0$. The corresponding embeddings thus coincide for both treatments of gauge field.

For $\hat{\psi} = \frac{\pi}{2}$ a difference arises in the radial embedding coordinate u. While in [8] C_u is negative for any choice of $\hat{\psi}$, here it becomes positive for $\cos 2\hat{\psi} < \frac{2}{9}$ which in particular is the case for $\hat{\psi} = \frac{\pi}{2}$. This changes the behaviour of the solution. In [8] for any angle $\hat{\psi}$ the function $u(\rho)$ is monotonically decreasing and hence corresponds to the first case in figure 1. Here, the function $u(\rho)$ assumes a relative maximum at an intermediate value

---

[4]Our definitions for B_ψ and C_ψ differs from the ones in [8]. To match the conventions there, we have to multiply our results by a factor $\frac{1}{u}$.

ρ if $\hat{\psi}$ fulfills

$$\cos 2\hat{\psi} < -\frac{8}{15} \ . \tag{5.19}$$

In particular this is the case for $\hat{\psi} = \frac{\pi}{2}$.

In figure 2 we compare the analytic solution with the numerical one for $F = 0$. The latter is based on the action

$$S = -\frac{T_7}{e^{\hat{\phi}}} \int d^4\xi \, d\Omega_3 \, d\rho \Big[\rho^3 \sqrt{1 + y'^2}$$
$$+ \frac{\rho^3 m^2 R^4}{36 r^4 \sqrt{1 + y'^2}} \Big(10\rho^2 + 14y^2 + 23\rho^2 y'^2 + y^2 y'^2 - 26\rho yy' - 24\sqrt{1 + y'^2}(y^2 - \rho yy') \Big) \Big] \ , \tag{5.20}$$

where $y = y^7$. This result replaces the corresponding one in [8]. We should stress that even if the numerical embeddings are obtained from the action (5.20) without making use of the expansion (4.5), they are not independent of it. The expansion has already been used in section 4 to obtain $F = 0$ which then enters the action (5.20).[5] For a complete independence of the expansion, one should solve the equations of motion for F and for the embedding coordinates directly as a coupled system. We refrain from this more complicated analysis, since the similarity of the numerical results and the exact solutions in figure 2 suggests that this should not change the numerical result significantly.

5.4 The $\mathcal{N} = 2$ case with $m_1 = m_2 = m$ and $m_3 = 0$

Since we have at hand the expression with generic masses, we can easily study the case $m_1 = m_2 = m$ and $m_3 = 0$ in which the Polchinski-Strassler background still preserves $\mathcal{N} = 2$ supersymmetries, but the embeddings oriented as shown in table 1 should break (part of) the supersymmetries. We denote them as the $\mathcal{N} = 2_\perp$ embeddings. Due to (5.16) and (5.17) the non-vanishing coefficients are given by

$$B_u = -\frac{\zeta^2 m^2 R^4}{54} \ , \qquad C_u = \frac{4\zeta^2 m^2 R^4}{81} \ , \qquad C_\psi^{\pm-} = \pm\frac{\zeta^2 m^2 R^4}{54} \ . \tag{5.21}$$

They do not depend on the unperturbed angular embedding coordinate $\hat{\psi}$. According to (5.10) and figure 1, the relations between B_u and C_u tell us that $u(\rho)$ is not monotonic,

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[5] An insertion of $F = 0$ into the equations of motion is equivalent to an insertion directly into the action.
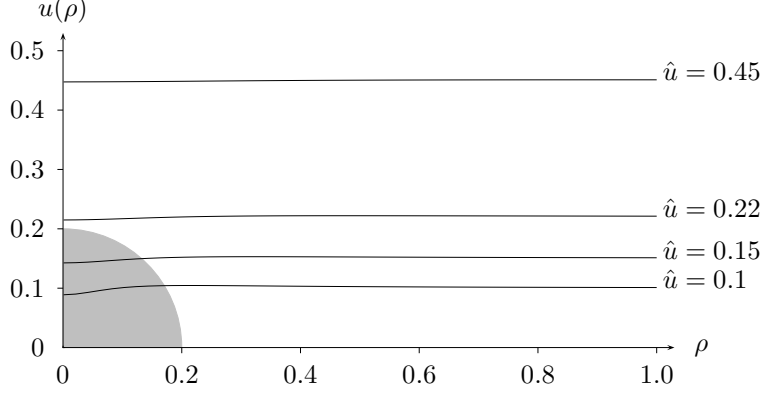
Figure 3: Radial embedding in the $\mathcal{N} = 2$ background with masses $m_1 = m_2 = m = 0.2$, $m_3 = 0$ and distinct boundary values $\hat{u}$. The grey quarter circle corresponds to $r \leq mR^2$, into which the background generating D3-branes are expected to expand. Lengths and masses are dimensionless and measured in units of R and R^{-1} respectively. The dimensionless boundary value $\hat{u}$ determines the dimensionful quark mass m_q according to $m_q = \frac{R}{2\pi\alpha'}\hat{u}$.

assuming a maximum at an intermediate value ρ. The radial embeddings are shown in figure 3. The angular embedding depends on all three angles in the four worldvolume coordinates y^a via the two $SO(4)$ spherical harmonics y_{+-} and y_{--} defined in (B.16).

5.5 The $\mathcal{N} = 1$ case with equal masses

A case of particular interest is the one of D7-brane embeddings into the $\mathcal{N} = 1$ Polchinski-Strassler background with equal masses. The non-vanishing coefficients of the solution in (5.9) are again found from (5.16) and (5.17). They read

$$B_u = -\frac{\zeta^2 m^2 R^4}{54}(3 - \cos 2\hat{\psi}) , \qquad C_u = \frac{\zeta^2 m^2 R^4}{27}(1 - 3\cos 2\hat{\psi}) , \qquad (5.22)$$

and

$$B_\psi = -\frac{\zeta^2 m^2 R^4}{54}\sin 2\hat{\psi} , \qquad C_\psi = \frac{2\zeta^2 m^2 R^4}{27}\sin 2\hat{\psi} , \qquad C_\psi^{+-} = \frac{\zeta^2 m^2 R^4}{27} . \qquad (5.23)$$

The function $u(\rho)$ ceases to be monotonic for
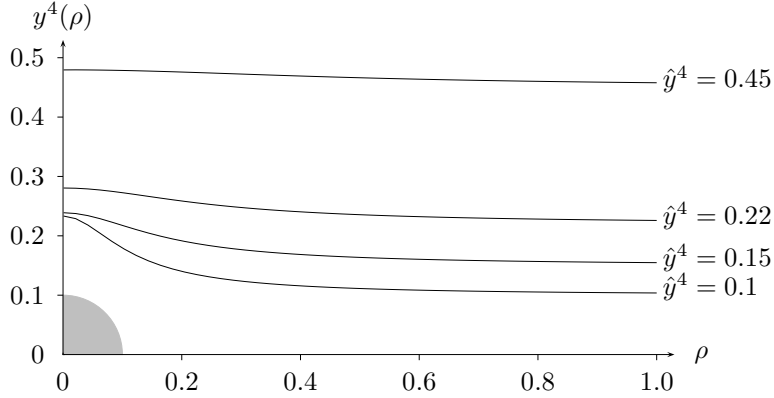
$$\cos 2\hat{\psi} < -\frac{1}{5} . \qquad (5.24)$$

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Figure 4: Embedding along y^4 in the $\mathcal{N} = 1$ background with masses $m_1 = m_2 = m_3 = m = 0.2$ and distinct boundary values $\hat{y}^4$. The grey quarter circle corresponds to $r \leq \frac{1}{2}mR^2$, into which the background generating D3-branes are expected to expand. Lengths and masses are dimensionless and measured in units of R and R^{-1} respectively. The dimensionless boundary value $\hat{y}^4$ determines the dimensionful quark mass m_q according to $m_q = \frac{R}{2\pi\alpha'}\hat{y}^4$.

As in the $\mathcal{N} = 2$ case, this in particular happens for $\hat{\psi} = \frac{\pi}{2}$. Embeddings with constant angular direction ψ do not exist at all in the $\mathcal{N} = 1$ case. However, the embeddings with $\hat{\psi} = 0$ or $\hat{\psi} = \frac{\pi}{2}$ are still peculiar, since they are directed along or respectively perpendicular to the principal axis with length determined by m_1 of the polarization ellipsoid of the D3-branes.

The explicit expression for the radial embedding coordinate in the case $\hat{\psi} = 0$ where $u = y^4$ and $2B_u = C_u$ reads

$$u = \hat{u}\left(1 + \frac{\zeta^2 m^2 R^4}{108}\left(\frac{1}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + \frac{1}{\hat{r}^2}\right)\right) , \tag{5.25}$$

while for $\hat{\psi} = \frac{\pi}{2}$ where $u = y^7$ we find

$$u = \hat{u}\left(1 + \frac{\zeta^2 m^2 R^4}{54}\left(\frac{1}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} - \frac{1}{\hat{r}^2}\right)\right) , \tag{5.26}$$

which has the property that $u(0) = \hat{u}$ as follows from (5.10) with the relation $2B_u = -C_u$ in this case. We have printed the corresponding y^4 embedding in figure 4 and the y^7
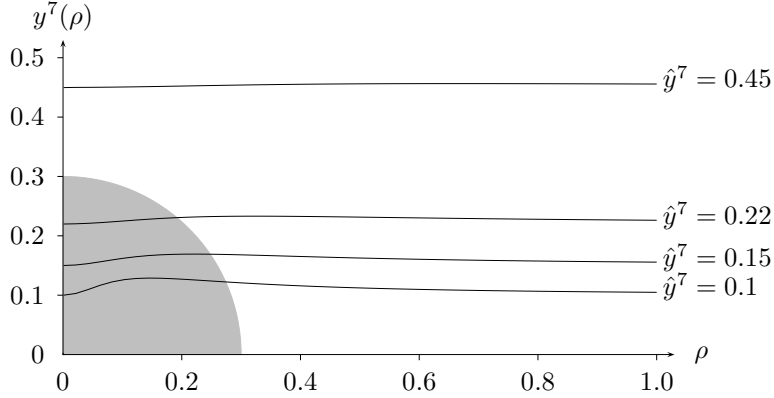
Figure 5: Embedding along y^7 in the $\mathcal{N} = 1$ background with masses $m_1 = m_2 = m_3 = 0.2$ and distinct boundary values $\hat{y}^7$. The grey quarter circle corresponds to $r \leq \frac{3}{2} m R^2$, into which the background generating D3-branes are expected to expand. Lengths and masses are dimensionless and measured in units of R and R^{-1} respectively. The dimensionless boundary value $\hat{y}^7$ determines the dimensionful quark mass m_q according to $m_q = \frac{R}{2\pi\alpha'} \hat{y}^7$.

embedding in figure 5. The background generating D3-branes extend in these directions with two different radii [8].

For the angular embeddings with $\hat{\psi} = 0$ or $\hat{\psi} = \frac{\pi}{2}$ we find $B_\psi = C_\psi = 0$ like in the corresponding $\mathcal{N} = 2$ cases. Only C_ψ^{+-} is non-zero. Defining polar coordinates for the four worldvolume coordinates y^a as in (B.17), the corresponding spherical harmonic y_{+-} only depends on two combinations of the three worldvolume angles. Up to the respective constant boundary values $\hat{\psi} = 0$ or $\hat{\psi} = \frac{\pi}{2}$, the angular embedding is identical for both cases. We find

$$\psi = \hat{\psi} + \frac{\zeta^2 m^2 R^4}{216} \left(\frac{2}{\rho^2} \left(1 - \frac{\hat{u}^2}{\rho^2} \ln \frac{\hat{r}^2}{\hat{u}^2} \right) - \frac{1}{\hat{r}^2} \right) y_{+-} \ . \tag{5.27}$$

In the 2-dimensional subplane given by $y^5 = y^8 = \frac{\rho}{\sqrt{2}} \cos \phi_1$, $y^6 = -y^9 = \frac{\rho}{\sqrt{2}} \sin \phi_1$ in the coordinates (B.17), in which according to (B.18) $y_{+-} = \cos 2\phi_1$, the angular embedding is shown in figure 6.
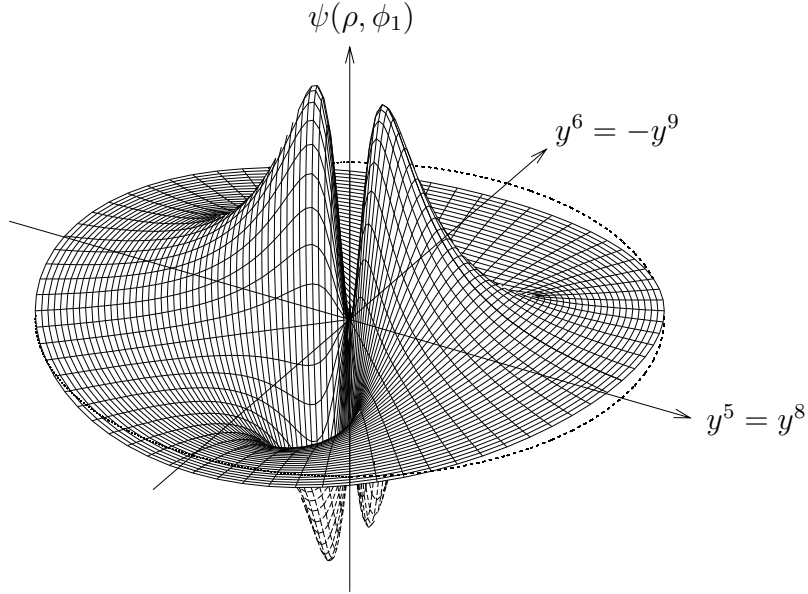
22

Figure 6: Form of the angular embedding in the $\mathcal{N} = 1$ background with masses $m_1 = m_2 = m_3 = m$ in the 2-dimensional subplane defined by $y^5 = y^8$, $y^6 = -y^9$.

5.6 Error estimates

With the expansion (4.5) we have found the regular analytic solutions (5.9) for the embedding coordinates. In the following we will analyse in which regimes these solutions with underlying action (5.1) are good approximations to the exact solutions which we can only find numerically from the corresponding action (3.6). We recall that by exact solutions we mean the exact solutions in the order $\mathcal{O}(m^2)$ Polchinski-Strassler background. One should keep in mind that even these solutions are limited to the regime in which a perturbative expansion of the background around $\text{AdS}_5 \times \text{S}^5$ is justified. The embeddings should avoid the deep interior of the space in which the extension of the background generating D3-brane sources becomes important. This requires $r \gtrsim mR^2$. The embeddings that are attracted by the origin of the space should have a boundary condition $\hat{u} \gtrsim mR^2$, while the ones that are repulsed can stay away from the interior of the space also for $\hat{u} \ll mR^2$.

To derive from the action (3.6) the result (5.1) we have neglected terms that are beyond linear order in $\tilde{y}^m$ and in the derivatives $\partial_a \tilde{y}^m$. This allows us to estimate an upper bound for the difference between the exact numerical embedding and the corre-

	$\mathcal{N}=2_\parallel$			$\mathcal{N}=2_\perp$		$\mathcal{N}=1$		
$\hat{\psi}$	0	$\frac{\pi}{2}$				0	$\frac{\pi}{2}$	
$\frac{B_u}{m^2R^4}$	-1	$-\frac{5}{3}$		$-\frac{1}{3}$		$-\frac{2}{3}$	$-\frac{4}{3}$	
$\frac{C_u}{m^2R^4}$	$-\frac{14}{9}$	$\frac{22}{9}$		$\frac{8}{9}$		$-\frac{4}{3}$	$\frac{8}{3}$	
$\frac{\rho_e}{\hat{u}}$	0	0	0.912	0	2.140	0	0	1.471
$\frac{\hat{u}\tilde{u}(\rho_e)}{m^2R^4}$	$\frac{4}{9}$	$\frac{1}{9}$	0.136	$-\frac{1}{36}$	0.011	$\frac{1}{3}$	0	0.072
$\frac{\rho_t}{\hat{u}}$	0.640	0.383	1.814	0.501	3.273	0.630	0.462	2.432
$\frac{\hat{u}^2 u'(\rho_t)}{m^2R^4}$	-0.226	0.045	-0.036	0.040	-0.002	-0.174	-0.090	-0.016

Table 2: The three different types of radial embeddings with their values of the constants B_u and C_u in units of m^2R^4 and the corresponding extrema ρ_e and turning points ρ_t in units of $\hat{u}$. The correction $\tilde{u}(\rho_e)$ is measured in units $\frac{m^2R^4}{\hat{u}}$ and the first derivative $u'(\rho_t)$ in units of $\frac{m^2R^4}{\hat{u}^2}$.

sponding analytic solution. It should be given by $\max\left(\left|\frac{\tilde{y}}{\hat{y}}\right|, |\partial\tilde{y}|\right)$. In table 2 we show the extrema ρ_e and turning points ρ_t of the radial embedding coordinates in the previously discussed special cases. The presented normalized expressions are independent of the explicit values of m, R and $\hat{u}$. To find the relative deviation from the exact result for the radial embedding on first has to select the maximum value of $\left|\frac{\tilde{u}(\rho_e)}{\hat{u}}\right|$ and $|u'(\rho_t)|$. For $\hat{u} = mR^2$ this directly gives the respective upper bound on the relative deviation. For $\hat{u} \neq mR^2$ we also have to restore the normalization by multiplying with $\frac{m^2R^4}{\hat{u}^2}$. For the y^4 embedding in the $\mathcal{N}=2_\parallel$ case the deviation is quite substantial with $\frac{4}{9} \simeq 44.4\%$. Doubling $\hat{u}$ brings it already down to 11.1%. For $\hat{u} = mR^2$ the y^7 embedding only deviates 13.6% from the exact solution. In all cases, the non-monotonic embeddings are much more accurately described by the analytic solution than the monotonic ones. Furthermore, the situation improves in the $\mathcal{N}=2_\perp$ case and in the $\mathcal{N}=1$ background with equal masses.

We should stress here that $\hat{u} = mR^2$ in general yields embeddings that run into a regime where the error from the perturbative expansion of the background itself should already be substantial. The corresponding analytic as well as the exact solutions should not be trusted carelessly. Only in the case of monotonically decreasing embeddings the exact solutions are substantially superior to the analytic ones, since they avoid the region of small r even for $\hat{u} \ll mR^2$ and hence can be trusted. The analytic solution does not

hold for $\hat{u} \ll mR^2$. The situation is different for the non-monotonic embeddings. The ones that avoid the region of small r obey $\hat{u} \gtrsim mR^2$ and thus are accurately described also by the analytic solution. A comparison with figure 2 furthermore suggests that the above described procedure provides quite appropriate estimates of the deviation.

6   Holographic renormalization

The non-constant boundary behaviour (5.10) of the found embeddings might imply the presence of a VEV for the fermion bilinear (quark condensate) in the dual gauge theory which would break supersymmetry. In the holographic gravity description the VEV is determined by varying the on-shell action w.r.t. the boundary value $\hat{u}$. This procedure requires holographic renormalization [35–37] to cancel the occurring divergences by appropriate counterterms. By also including appropriate finite counterterms the renormalized on-shell action can be made vanishing. In particular, a quark condensate is hence absent and our found embeddings of the form (5.9) are at least consistent with supersymmetry. Surprisingly, the procedure works for the action (5.1) independent of the concrete values for the introduced constants. Therefore, also the case with an alternative relative sign choice between the DBI and CS action is covered, implying that the procedure does not provide further information for finally fixing this sign.

In appendix C we derive the explicit form of the expanded action (5.1), which is then given by (C.7). The terms of relevance for the holographic renormalization procedure read

$$
S = -\frac{T_7}{e^{\hat{\phi}}} \int d\xi^4 \, d\Omega_3 \, d\rho \rho^3 \Big[1 + \frac{B_u}{2} \frac{\hat{u}^2}{\hat{r}^4}
$$
$$
+ \frac{\zeta^2}{216} \hat{Z} \Big(\frac{5}{3} M^2 (\hat{r}^2 + \hat{u}^2) + 3m_1((m_2 + m_3)y_{++} - (m_2 - m_3)y_{-+})\rho^2 \Big) \Big] ,
$$
$$
\tag{6.1}
$$

where we remind that M^2 is the sum of all three mass squares as defined in (2.12). We transform to a new coordinate χ that parameterizes the radial direction. The relations read

$$
\hat{r} = \frac{1}{\sqrt{\chi}} , \qquad \rho^2 = \frac{1}{\chi} - \hat{u}^2 , \qquad \rho \, d\rho = -\frac{d\chi}{2\chi^2} , \qquad \partial_\rho = -2\rho\chi^2 \partial_\chi . \tag{6.2}
$$

Performing the radial integration over the interval $\varepsilon \leq \chi \leq \frac{1}{\hat{u}^2}$, the regularized on-shell

action is given by

$$S_{\text{reg}} = -\frac{T_7}{2\,e^{\hat{\phi}}} \int d\xi^4 \, d\Omega_3 \left[\frac{1}{2\varepsilon^2} + \frac{\hat{u}^4}{2} - \frac{\hat{u}^2}{\varepsilon} - B_u \frac{\hat{u}^2}{2} (\ln \varepsilon \hat{u}^2 + 1 - \varepsilon \hat{u}^2) \right. $$
$$\left. + \frac{\zeta^2 R^4}{216} \frac{5}{3} M^2 \left(\frac{1}{\varepsilon} - 2\hat{u}^2 + \varepsilon \hat{u}^4 \right) \right] . \tag{6.3}$$

From (5.9) we derive the solution for the radial embedding coordinate in the variable χ. It is given by

$$u = \hat{u} - \frac{\hat{u}}{8} \chi \left(- B_f \frac{2}{1 - \chi \hat{u}^2} \ln \chi \hat{u}^2 + C_f + C_f^I \left(1 - \frac{2}{1 - \chi \hat{u}^2} - \frac{2\chi \hat{u}^2}{(1 - \chi \hat{u}^2)^2} \ln \chi \hat{u}^2 \right) y_I \right) . \tag{6.4}$$

We evaluate the above relation at $\chi = \varepsilon$, expand it up to order $\mathcal{O}(\varepsilon^2)$, and invert it to express the boundary value $\hat{u}$ in terms of the value $u_\varepsilon = u(\varepsilon)$. The inverted relation then reads

$$\hat{u} = u_\varepsilon \left(1 + \frac{\varepsilon}{8} \left(- 2B_u (1 + \varepsilon u_\varepsilon^2) \ln \varepsilon u_\varepsilon^2 + C_u - C_u^I (1 + 2\varepsilon u_\varepsilon^2 (1 + \ln \varepsilon u_\varepsilon^2)) y_I \right) \right) . \tag{6.5}$$

This result is inserted into the regularized on-shell action (6.3). The terms that contain a single spherical harmonic y_I drop out when the angle integration is performed. Thus we obtain

$$S_{\text{reg}} = -\frac{T_7}{e^{\hat{\phi}}} \frac{\Omega_3}{2} \int d\xi^4 \left[\frac{1}{2\varepsilon^2} + \frac{u_\varepsilon^4}{2} - \frac{u_\varepsilon^2}{\varepsilon} - (2B_u + C_u) \frac{u_\varepsilon^2}{4} (1 - \varepsilon u_\varepsilon^2) \right. $$
$$\left. + \frac{\zeta^2 R^4}{216} \frac{5}{3} M^2 \left(\frac{1}{\varepsilon} - 2u_\varepsilon^2 + \varepsilon u_\varepsilon^4 \right) \right] , \tag{6.6}$$

where Ω_3 is the volume of the unit S^3. In contrast to (6.3), which is a functional of the boundary value $\hat{u}$, the above result depends on the data u_ε at the regulator hypersurface at $\chi = \varepsilon$. By this change of variables the logarithmic term that is present in (6.3) cancels out. With the local counterterm action given by

$$S_{\text{ct}} = \frac{T_7}{e^{\hat{\phi}}} \frac{\Omega_3}{2} \int d\xi^4 \left[\frac{1}{2\varepsilon^2} + \frac{u_\varepsilon^4}{2} - \frac{u_\varepsilon^2}{\varepsilon} - (2B_u + C_u) \frac{u_\varepsilon^2}{4} + \frac{\zeta^2 R^4}{216} \frac{5}{3} M^2 \left(\frac{1}{\varepsilon} - 2u_\varepsilon^2 \right) \right] \tag{6.7}$$

we can then make the subtracted action $S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}}$ vanish. To this purpose we have also included finite counterterms in the above expression. The explicit form of the

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combination $2B_u + C_u$ found from (B.21) is given by

$$2B_u + C_u = \frac{\zeta^2 R^4}{18}\Big((m_2 + m_3)^2\tau + (m_2 - m_3)^2(\gamma - \delta)$$
$$+ m_1^2(\gamma + \delta - \tau) + 2m_2m_3(\gamma - \delta - \tau)(1 - \cos 2\hat{\psi}) - \frac{10}{9}M^2\Big) .$$
$$(6.8)$$

The counter term action is then explicitly given by

$$S_{\text{ct}} = \frac{T_7}{e^{\hat{\phi}}}\frac{\Omega_3}{2}\int d\xi^4 \Big[\frac{1}{2\varepsilon^2} + \frac{u_\varepsilon^4}{2} - \frac{u_\varepsilon^2}{\varepsilon} + \frac{\zeta^2 R^4}{216}\frac{5}{3\varepsilon}M^2$$
$$- \frac{\zeta^2 R^4}{72}\Big((m_2 + m_3)^2\tau + (m_2 - m_3)^2(\gamma - \delta)$$
$$+ m_1^2(\gamma + \delta - \tau) + 2m_2m_3(\gamma - \delta - \tau)(1 - \cos 2\hat{\psi})\Big)u_\varepsilon^2\Big] .$$
$$(6.9)$$

For any values for the constants and mass parameters that enter the action (5.1) we can therefore obtain $S_{\text{sub}} = 0$ at least up to order $\mathcal{O}(m^2)$ and hence show that no quark condensate can be present up to this order.

We should remark that in the $\mathcal{N} = 2$ Polchinski-Strassler background with the parameters given by (3.7) and (5.4) our result (6.9) should reduce to the one found in [8]. However, our equation (F.1) in [8] contains an error. It only affects the embedding with $\hat{\psi} = \frac{\pi}{2}$, since it is caused by a wrong sign in front of a term which is proportional to $1 - \cos 2\hat{\psi}$. To correct this mistake, one has to replace $-(\frac{1}{3} + \cos 2\hat{\psi})$ in the second line of (F.1) by $-(\frac{7}{3} - \cos 2\hat{\psi})$. In equation (F.3) one then has to set $c_0 = -\frac{10}{3}$. This mistake has no further effect, since the statement $c_0 + c_1 - \frac{5}{3} = 0$, which is essential for the procedure to succeed, is in fact only fulfilled by the corrected numerical value. The finite counterterm then depends on $\hat{\psi}$, as is also seen from the above result (6.9).

7   Conclusions

In this paper we have analyzed the embedding of D7-brane probes into the Polchinski-Strassler background at order $\mathcal{O}(m^2)$, keeping the three mass perturbation parameters general. To this order we have seen that all embeddings are consistent with a vanishing gauge field strength $F = 0$ on their worldvolumes. Thereby, the expansion of the embedding coordinates y^m into a constant unperturbed embedding $\hat{y}^m$ in $\text{AdS}_5 \times \text{S}^5$ and a non-constant correction $\tilde{y}^m$ of order $\mathcal{O}(m^2)$ decoupled the differential equations for F

and y^m. This expansion resembled the perturbative expansion in which the known part of the Polchinski-Strassler background itself is given. It also allowed us to find analytic solutions for the expanded embedding.

If the additional constant γ introduced into the action assumes its value $\gamma = 1$, the radial embedding coordinate u is a function of only the worldvolume radial direction ρ for all values of the mass parameters and for all choices of the embedding angle $\hat{\psi}$. The angular embedding coordinate ψ itself depends on the three worldvolume angles if the mass parameter m_1 associated to the embedding directions z^1, $\bar{z}^1$ is non-zero. For $\gamma = 1$ the angular dependence is encoded in only two $l = 2$ $SO(4)$ spherical harmonics, and their coefficients do not depend on the boundary value $\hat{\psi}$ of the embedding angle. Moreover, the angular dependence reduces to only one spherical harmonic if the two masses m_a, $a = 2, 3$ that correspond to the worldvolume directions z^a, $\bar{z}^a$ are equal. A complete independence from the worldvolume angles as found in the $\mathcal{N} = 2$ case [8] cannot be reached in the Polchinski-Strassler case if all masses are different from zero, even if they are equal. This would require that the parameters in the action (5.1) fulfilled $\gamma + \delta + \tau = 1$, such that with $\gamma = 1$ embeddings with arbitrary $\hat{\psi}$ would not depend on the worldvolume angles. If $\gamma + \delta + \tau = 1$ but $\gamma \neq 1$ at least the embeddings with $\hat{\psi} = 0, \frac{\pi}{2}$ became angle independent. The angle dependence of the embeddings is understandable if one remembers that for $m_1 = m_2 = m_3 = m$ the background generating D3-branes are polarized into an ellipsoid with distinct lengths of its principal axes. This breaks the $SO(4)$ rotational symmetry in the worldvolume directions of the embedded D7-brane [8]. Surprisingly, this does not affect the radial embedding coordinate u which with $\gamma = 1$ in all cases only depends on ρ.

It would be interesting to find an interpretation for the condition $\gamma + \delta + \tau = 0$ for which angle independent embeddings can be found. The above given relation might be fulfilled in a more symmetric background in which the D3-branes are polarized not into an ellipsoid but into a sphere. In this context one could analyze the polarization in presence of a non-vanishing gaugino mass [11]. Furthermore, it appears to be interesting to study the embedding of a stack of coincident D7-branes with non-vanishing worldvolume instantons along the lines of [21, 38, 39]. The backreaction of the instanton gauge field strength on the embeddings could also influence their worldvolume angle dependence. There might exist particular cases in which the embeddings do not depend on these angles.

We have then discussed the monotony properties of the analytic radial solutions $u(\rho)$ and found four types of embeddings. For all of these the radial coordinate in six dimensions $r(\rho)$ is a monotonically increasing function at least for sufficiently large boundary value $\hat{u}$, such that the embeddings are physical [24].

With $F = 0$ we have revisited the y^7 embedding in the $\mathcal{N} = 2$ case with our modified

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treatment of the gauge field. This changed the monotonically decreasing $y^7(\rho)$ found in [8] into a non-monotonic one that still led to a monotonically increasing $r(\rho)$. We compared the analytic solution with the numerical exact solution and found agreement.

Furthermore, we have discussed in brief the case of a D7-brane probe with a different orientation in the $\mathcal{N} = 2$ background and the y^4 and y^7 embeddings in the $\mathcal{N} = 1$ case with equal masses. We then proposed error estimates for the corresponding analytic solutions.

In a last step we have applied the method of holographic renormalization to the action (5.1) of a D7-brane probe. In the general case this demonstrated that the embeddings did not induce a non-vanishing quark condensate in the dual boundary theory, since the subtracted action could be made vanish by adding appropriate finite counterterms.

Our final remark concerns the meson spectra. We believe that qualitatively for all the discussed embeddings the corresponding spectra will show mass gaps, and the squared meson mass $M^2(m_{\mathrm{q}}^2)$ should be a (nearly) linear function of the squared quark mass m_{q}^2. In case of the $\mathcal{N} = 2_{\parallel}$ embeddings we have already given indications for this behaviour in [8].

Acknowledgements

A Generalization to arbitrary masses

Here we present the necessary expressions to compute the equations of motion and the action explicitly. They are the generalizations to three arbitrary masses of the corresponding expressions in the $\mathcal{N} = 2$ Polchinski-Strassler background with $m_1 = 0$, $m_2 = m_3 = m$ computed in [8], where all necessary conventions can be found. We again work in the complex basis (2.7). Inserting the components (2.8) of T_3 in this basis into

(2.4), we find for the components of $\tilde{C}_2$ and B the explicit expressions

$$
\begin{aligned}
\tilde{C}_{pq} &= \mathrm{e}^{-\hat{\phi}}\frac{\zeta}{6}Z\epsilon_{rpq}m_r\bar{z}^r \ , & B_{pq} &= -i\frac{\zeta}{6}Z\epsilon_{rpq}m_r\bar{z}^r \ , \\
\tilde{C}_{p\bar{q}} &= \mathrm{e}^{-\hat{\phi}}\frac{\zeta}{6}Z\epsilon_{rpq}(m_p\bar{z}^r + m_q z^r) \ , & B_{p\bar{q}} &= i\frac{\zeta}{6}Z\epsilon_{rpq}(m_p\bar{z}^r - m_q z^r) \ , \\
\tilde{C}_{\bar{p}q} &= \mathrm{e}^{-\hat{\phi}}\frac{\zeta}{6}Z\epsilon_{rpq}(m_q\bar{z}^r + m_p z^r) \ , & B_{\bar{p}q} &= i\frac{\zeta}{6}Z\epsilon_{rpq}(m_q\bar{z}^r - m_p z^r) \ , \\
\tilde{C}_{\bar{p}\bar{q}} &= \mathrm{e}^{-\hat{\phi}}\frac{\zeta}{6}Z\epsilon_{rpq}m_r z^r \ , & B_{\bar{p}\bar{q}} &= i\frac{\zeta}{6}Z\epsilon_{rpq}m_r z^r \ .
\end{aligned}
\tag{A.1}
$$

If the D7-brane is embedded as shown in table 1, the directions z^a, $\bar{z}^a$ $a = 2, 3$ run along its worldvolume, while z^m, $\bar{z}^m$, $m = 1$ are perpendicular to it. The following expressions are defined in the four flat parallel directions, and hence all inner products (denoted by $\cdot$) and Hodge star operators (denoted by $\star_4$) are understood to be the ones in the 4-dimensional flat space spanned by z^a, $\bar{z}^a$, $a = 2, 3$.

With the definition of the sum of the squared masses in (2.12) the inner products in four dimensions thus become

$$
\begin{aligned}
\tilde{C}_2 \cdot \tilde{C}_2 &= \tilde{C}_{ab}\tilde{C}_{\bar{a}\bar{b}} + \tilde{C}_{a\bar{b}}\tilde{C}_{\bar{a}b} = \mathrm{e}^{-2\hat{\phi}}\frac{\zeta^2}{18}Z^2(M^2 z^m\bar{z}^m + m_a m_b(z^m z^m + \bar{z}^m\bar{z}^m)) \ , \\
B \cdot B &= B_{ab}B_{\bar{a}\bar{b}} + B_{a\bar{b}}B_{\bar{a}b} = \frac{\zeta^2}{18}Z^2(M^2 z^m\bar{z}^m - m_a m_b(z^m z^m + \bar{z}^m\bar{z}^m))
\end{aligned}
\tag{A.2}
$$

and

$$
\begin{aligned}
\tilde{C}_2 \cdot \star_4\tilde{C}_2 &= -\tilde{C}_{ab}\tilde{C}_{\bar{a}\bar{b}} + \tilde{C}_{a\bar{a}}\tilde{C}_{b\bar{b}} + \tilde{C}_{a\bar{b}}\tilde{C}_{\bar{a}b} \\
&= -\mathrm{e}^{-2\hat{\phi}}\frac{\zeta^2}{18}Z^2((m_m^2 - m_a^2 - m_b^2)z^m\bar{z}^m - m_a m_b(z^m z^m + \bar{z}^m\bar{z}^m)) \ , \\
B \cdot \star_4 B &= -B_{ab}B_{\bar{a}\bar{b}} + B_{a\bar{a}}B_{b\bar{b}} + B_{a\bar{b}}B_{\bar{a}b} \\
&= -\frac{\zeta^2}{18}Z^2((m_m^2 - m_a^2 - m_b^2)z^m\bar{z}^m + m_a m_b(z^m z^m + \bar{z}^m\bar{z}^m)) \ ,
\end{aligned}
\tag{A.3}
$$

where on the r.h.s. of the above expressions the indices $a, b, m \in \{1, 2, 3\}$ take fixed distinct values.

A combination that appears in the action (5.1) and in the equations of motion (5.5)

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for the embedding coordinates is then determined as

$$\frac{1}{2}Z^{-1}\big((\alpha - \beta \star_4)B \cdot B + \tau\, \mathrm{e}^{2\hat\phi} \star_4 \tilde C_2 \cdot \tilde C_2\big)$$

$$= \frac{\zeta^2}{36}Z\big((\alpha + \beta - \tau)m_m^2 + (\alpha - \beta + \tau)(m_a^2 + m_b^2))z^m\bar z^m \tag{A.4}$$

$$- (\alpha - \beta - \tau)m_a m_b(z^m z^m + \bar z^m \bar z^m)\big)\ ,$$

where $\star_4$ is understood to act on the first form on its right.

Furthermore, one needs similar expressions where not all components are summed. They read

$$Z^{-1}\big((\gamma - \delta\star_4)(B_{ab}B_{m\bar b} + B_{a\bar b}B_{mb}) + \tau\, \mathrm{e}^{2\hat\phi} \star_4(\tilde C_{ab}\tilde C_{m\bar b} + \tilde C_{a\bar b}\tilde C_{mb})\big)$$

$$= -\frac{\zeta^2}{36}Z\big(((\gamma - \delta + \tau)m_a^2 + (\gamma + \delta - \tau)m_m^2)\bar z^a \bar z^m$$

$$- m_b((\gamma + \delta + \tau)m_m z^a \bar z^m + (\gamma - \delta - \tau)m_a \bar z^a z^m)\big)\ ,$$

$$Z^{-1}\big((\gamma - \delta\star_4)(B_{\bar ab}B_{m\bar b} + B_{\bar a\bar b}B_{mb}) + \tau\, \mathrm{e}^{2\hat\phi} \star_4(\tilde C_{\bar ab}\tilde C_{m\bar b} + \tilde C_{\bar a\bar b}\tilde C_{mb})\big) \tag{A.5}$$

$$= -\frac{\zeta^2}{36}Z\big(2\gamma m_a m_m \bar z^a z^m - (\gamma - \delta - \tau)m_b(m_a z^a z^m + m_m \bar z^a \bar z^m)$$

$$+ (\gamma - \delta + \tau)m_b^2 z^a \bar z^m\big)\ ,$$

where on the l.h.s. one first has to act with $\star_4$ on the first tensor on its right, and then extract the required components. While on the l.h.s. a sum over b is understood and a, m take fixed values, on the r.h.s. all indices a, b, m take the corresponding fixed distinct values.

The tensors (2.11) for the corrected metric (2.10) read in complex coordinates

$$I_{pq} = -\frac{\bar z^p \bar z^q}{10z\bar z}\ , \qquad I_{p\bar q} = \frac{1}{5}\Big(\delta_{p\bar q} - \frac{\bar z^p z^q}{2z\bar z}\Big)\ , \tag{A.6}$$

and

$$W_{pq} = \frac{1}{4M^2 z\bar z}(2\delta_{p\bar q}m_p m_r \bar z^r \bar z^r - (m_p^2 + m_q^2)\bar z^p \bar z^q) + \frac{\bar z^p \bar z^q}{10z\bar z}\ ,$$

$$W_{p\bar q} = \frac{1}{20}\Big(\delta_{p\bar q} - 3\frac{\bar z^p z^q}{z\bar z}\Big) + \frac{1}{4M^2 z\bar z}((m_p^2 + m_q^2)\bar z^p z^q - 2m_p m_q z^p \bar z^q)\ , \tag{A.7}$$

where we have introduced $z\bar z = z^p \bar z^p$ to abbreviate the expression summed over p. The remaining components are obtained by complex conjugation from the above expressions.

Taking the traces of the corrections in (2.10) w.r.t. to the four-dimensional subspace, i.e. summing over $a = 2, 3$, thereby using that in the complex basis the radii defined in (5.6) become

$$r = \sqrt{2z\bar{z}} = \sqrt{\rho^2 + u^2} \ , \qquad \rho = \sqrt{2z^a \bar{z}^a} \ , \qquad u = \sqrt{2z^m \bar{z}^m} \ , \qquad (A.8)$$

we find

$$\tilde{g}_{aa} = \frac{1}{10}(6p + 10q - w) + \frac{u^2}{r^2}\frac{1}{10}(2p - 10q + 3w) \ . \qquad (A.9)$$

Using that according to (2.14) the trace over the first four directions is given by $Z\tilde{g}_{\mu\mu} = 4h_0 Z = q - p$, the required combination of traces becomes

$$Z^{-\frac{1}{2}}(Z\tilde{g}_{\mu\mu} + \tilde{g}_{aa}) = \frac{1}{10R^2}\left(\rho^2(-4p + 20q - \omega) + u^2(-2p + 10q + 2\omega)\right) \ . \qquad (A.10)$$

Furthermore, the required off-diagonal elements read

$$\begin{aligned}
\tilde{g}_{am} &= \frac{1}{10z\bar{z}}(5q - p + w)\bar{z}^a \bar{z}^m - \frac{w}{4M^2 z\bar{z}}(m_a^2 + m_m^2)\bar{z}^a \bar{z}^m \ , \\
\tilde{g}_{\bar{a}m} &= \frac{1}{20z\bar{z}}(10q - 2p - 3w)z^a \bar{z}^m + \frac{w}{4M^2 z\bar{z}}((m_a^2 + m_m^2)z^a \bar{z}^m - 2m_a m_m \bar{z}^a z^m) \ ,
\end{aligned} \qquad (A.11)$$

where the missing combinations are obtained by complex conjugation.

B Perturbative expansion of the equations of motion

In this section we derive the explicit form of the equations of motion (5.5) for the expanded embedding coordinates (4.5). In the complex basis (2.7) the equations of motion (5.5) with $F = 0$ are given by

$$\begin{aligned}
&2\partial_a \partial_{\bar{a}} \tilde{z}^m + \partial_a\left(Z^{-\frac{1}{2}}\tilde{g}_{m\bar{a}}\right) + \partial_{\bar{a}}\left(Z^{-\frac{1}{2}}\tilde{g}_{ma}\right) \\
&\quad + \partial_a\left(Z^{-1}\left((\gamma - \delta\star_4)(B_{\bar{a}b}B_{m\bar{b}} + B_{\bar{a}\bar{b}}B_{mb}) + \tau\, \mathrm{e}^{2\hat{\phi}}((\star_4\tilde{C})_{\bar{a}b}\tilde{C}_{m\bar{b}} + (\star_4\tilde{C})_{\bar{a}\bar{b}}\tilde{C}_{mb})\right)\right) \\
&\quad + \partial_{\bar{a}}\left(Z^{-1}\left((\gamma - \delta\star_4)(B_{ab}B_{m\bar{b}} + B_{a\bar{b}}B_{mb}) + \tau\, \mathrm{e}^{2\hat{\phi}}((\star_4\tilde{C})_{ab}\tilde{C}_{m\bar{b}} + (\star_4\tilde{C})_{a\bar{b}}\tilde{C}_{mb})\right)\right) \\
&= \frac{\partial}{\partial\tilde{z}^m}\left(\tilde{\phi} + \frac{1}{2}Z^{\frac{1}{2}}\tilde{g}_{\mu\mu} + \frac{1}{2}Z^{-\frac{1}{2}}\tilde{g}_{aa} + \frac{1}{2}Z^{-1}\left((\alpha - \beta\star_4)B \cdot B + \tau\, \mathrm{e}^{2\hat{\phi}} \star_4 \tilde{C}_2 \cdot \tilde{C}_2\right)\right)\Big|_{\tilde{z}^m = \bar{\tilde{z}}^m = 0}
\end{aligned}$$
$$(B.1)$$

and by its complex conjugate. The individual expressions that enter the above equation are given by the derivatives of the results computed in Appendix A. From (A.4) we find with the definition of ρ, u and r in (5.6) and in (A.8)

$$
\begin{aligned}
\partial_m \Big(\frac{1}{2} Z^{-1} \big((\alpha - \beta \star_4) B \cdot B + \tau \, e^{2\hat{\phi}} \star_4 \tilde{C}_2 \cdot \tilde{C}_2 \big) \Big) \\
= \frac{\zeta^2}{36} Z \Big((\alpha + \beta - \tau) m_m^2 + (\alpha - \beta + \tau)(m_a^2 + m_b^2) \Big) \Big(1 - 2\frac{u^2}{r^2} \Big) \bar{z}^m \\
- 2(\alpha - \beta - \tau) m_a m_b \Big(\Big(1 - \frac{u^2}{r^2} \Big) z^m - 2\frac{(\bar{z}^m)^3}{r^2} \Big) \Big) \;,
\end{aligned} \tag{B.2}
$$

where on the r.h.s. the indices a, b, and m take fixed distinct values. The divergence of (A.5) reads

$$
\begin{aligned}
\partial_a \big(Z^{-1} \big((\gamma - \delta \star_4)(B_{\bar{a}b} B_{m\bar{b}} + B_{\bar{a}\bar{b}} B_{mb}) + \tau \, e^{2\hat{\phi}} \star_4 (\tilde{C}_{\bar{a}b} \tilde{C}_{m\bar{b}} + \tilde{C}_{\bar{a}\bar{b}} \tilde{C}_{mb})) \big) + (a \leftrightarrow \bar{a}) \\
= \frac{\zeta^2}{18} Z \Big(- \big(((\gamma - \delta + \tau)(m_a^2 + m_b^2) + (\gamma + \delta - \tau) m_m^2) \bar{z}^m - 2(\gamma - \delta - \tau) m_b m_a z^m \big) \frac{u^2}{r^2} \\
+ \frac{2}{r^2} m_m \big(2\gamma (m_a \bar{z}^a \bar{z}^a + m_b \bar{z}^b \bar{z}^b) z^m \\
- (m_b((\gamma + \delta + \tau) z^a z^a + (\gamma - \delta - \tau) \bar{z}^a \bar{z}^a) \\
+ m_a((\gamma + \delta + \tau) z^b z^b + (\gamma - \delta - \tau) \bar{z}^b \bar{z}^b)) \bar{z}^m \big) \Big) \;,
\end{aligned} \tag{B.3}
$$

where on the l.h.s. we have abbreviated the second term which is found from the first one by exchanging the summation indices a and $\bar{a}$. While a and b are summed over on the l.h.s., on the r.h.s. a, b, m take fixed distinct values. The gradient of the dilaton as given in (2.15) becomes in complex coordinates

$$
\begin{aligned}
\partial_m \tilde{\phi} = \frac{\zeta^2}{18} Z \Big(- \frac{2}{r^2} (m_a m_b \bar{z}^m \bar{z}^m + m_a m_m (z^b z^b + \bar{z}^b \bar{z}^b) + m_b m_m (z^a z^a + \bar{z}^a \bar{z}^a)) \bar{z}^m \\
+ m_a m_b \Big(1 - \frac{u^2}{r^2} \Big) z^m \Big) \;,
\end{aligned} \tag{B.4}
$$

where a, b, m take fixed distinct values. The derivative of the subtraces of the corrections to the metric in (A.10) become

$$
\frac{1}{2} \partial_m \big(Z^{-\frac{1}{2}} (Z \tilde{g}_{\mu\mu} + \tilde{g}_{aa}) \big) = \frac{1}{5R^2} \Big(3p - 15q + 2w - (2p - 10q + 3w) \frac{u^2}{r^2} \Big) \bar{z}^m \;. \tag{B.5}
$$

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Finally, the derivatives of the off-diagonal elements of the corrections to the metric in (A.11) are found to be given by

$$\partial_a(Z^{-\frac{1}{2}}\tilde{g}_{\bar{a}m}) + \partial_{\bar{a}}(Z^{-\frac{1}{2}}\tilde{g}_{am})$$
$$= \frac{1}{5R^2}(20q - 4p - \omega)\frac{u^2}{r^2}\bar{z}^m + \frac{4\omega}{M^2R^2r^2}m_m(m_a\bar{z}^a\bar{z}^a + m_b\bar{z}^b\bar{z}^b)z^m .$$
(B.6)

While a is a summation index on the l.h.s., a, b and m take fixed distinct values on the r.h.s. Inserting the above equations into (B.1), we obtain

$$2\partial_a\partial_{\bar{a}}\tilde{\bar{z}}^m = -\frac{2}{r^2}\left(\frac{\zeta^2}{9}Z\gamma + \frac{2\omega}{M^2R^2}\right)m_m(m_a\bar{z}^a\bar{z}^a + m_b\bar{z}^b\bar{z}^b)z^m$$
$$+ \frac{\zeta^2}{36}Z\left(\frac{4}{r^2}m_m\left(m_b((\gamma + \delta + \tau - 1)z^a z^a + (\gamma - \delta - \tau - 1)\bar{z}^a\bar{z}^a)\right.\right.$$
$$+ m_a((\gamma + \delta + \tau - 1)z^b z^b + (\gamma - \delta - \tau - 1)\bar{z}^b\bar{z}^b))\bar{z}^m$$
$$+ m_m^2\left(\alpha + \beta - \tau - 2(\alpha + \beta - \gamma - \delta)\frac{u^2}{r^2}\right)\bar{z}^m$$
$$+ (m_a^2 + m_b^2)\left(\alpha - \beta + \tau - 2(\alpha - \beta - \gamma + \delta)\frac{u^2}{r^2}\right)\bar{z}^m$$
$$- 2m_am_b\left(\left(\alpha - \beta - \tau - 1 - (\alpha - \beta - 2(\gamma - \delta) + \tau - 1)\frac{u^2}{r^2}\right)z^m\right.$$
$$\left.\left.- 2(\alpha - \beta - \tau - 1)\frac{(\bar{z}^m)^3}{r^2}\right)\right)$$
$$+ \frac{1}{5R^2}\left(3p - 15q + 2\omega + (2p - 10q - 2\omega)\frac{u^2}{r^2}\right)\bar{z}^m ,$$
(B.7)

where on the l.h.s. a is a summation index. On the r.h.s. a, b and m take fixed distinct values, and the expressions have to be evaluated using the unperturbed embedding coordinates $\hat{z}^m$ and $\hat{\bar{z}}^m$ as required by (B.1).

We use the explicit values for p, q and ω given in (2.13) to compute the combinations

$$3p - 15q + 2\omega = -\frac{5\zeta^2}{27}M^2R^2Z , \qquad 2p - 10q - 2\omega = \frac{5\zeta^2}{81}M^2R^2Z .$$
(B.8)

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They are used to obtain the result

$$
2\partial_a\partial_{\bar a}\bar{\hat{z}}^m = \frac{\zeta^2}{36}\hat Z\Big(\frac{4}{\hat r^2}m_m\big(2(1-\gamma)(m_a\bar z^a\bar z^a + m_b\bar z^b\bar z^b)\hat z^m
$$
$$
+ (m_b((\gamma+\delta+\tau-1)z^a z^a + (\gamma-\delta-\tau-1)\bar z^a\bar z^a)
$$
$$
+ m_a((\gamma+\delta+\tau-1)z^b z^b + (\gamma-\delta-\tau-1)\bar z^b\bar z^b))\bar{\hat z}^m\big)
$$
$$
+ m_m^2\Big(\alpha+\beta-\tau-\frac{4}{3} - 2\Big(\alpha+\beta-\gamma-\delta-\frac{2}{9}\Big)\frac{\hat u^2}{\hat r^2}\Big)\bar{\hat z}^m
$$
$$
+ (m_a^2 + m_b^2)\Big(\alpha-\beta+\tau-\frac{4}{3} - 2\Big(\alpha-\beta-\gamma+\delta-\frac{2}{9}\Big)\frac{\hat u^2}{\hat r^2}\Big)\bar{\hat z}^m
$$
$$
- 2m_a m_b\Big(\Big(\alpha-\beta-\tau-1 - (\alpha-\beta-2(\gamma-\delta)+\tau-1)\frac{\hat u^2}{\hat r^2}\Big)\hat z^m
$$
$$
- 2(\alpha-\beta-\tau-1)\frac{(\bar{\hat z}^m)^3}{\hat r^2}\Big)\Big) \ ,
$$

(B.9)

where the quantities that carry a 'hat' are related to or respectively evaluated with the unperturbed part of the embedding. Multiplying both sides with $\hat z^m$, adding and subtracting the complex conjugate of the result, and making use of the decomposition

$$
f\phi + g\bar\phi = \frac{f+g}{2}(\phi+\bar\phi) + \frac{f-g}{2}(\phi-\bar\phi) \ ,
$$

(B.10)

valid for arbitrary f, g and ϕ, we obtain the two equations

$$
2(\hat z^m\partial_a\partial_{\bar a}\bar{\hat z}^m + \bar{\hat z}^m\partial_a\partial_{\bar a}\hat z^m)
$$
$$
= \frac{\zeta^2}{36}\hat Z\hat u^2\Big(\frac{2}{\hat r^2}(1-\gamma)m_m\Big((m_a(z^a z^a + \bar z^a\bar z^a) + m_b(z^b z^b + \bar z^b\bar z^b))\Big(\frac{\hat z^m}{\bar{\hat z}^m} + \frac{\bar{\hat z}^m}{\hat z^m}\Big)
$$
$$
- (m_a(z^a z^a - \bar z^a\bar z^a) + m_b(z^b z^b - \bar z^b\bar z^b))\Big(\frac{\hat z^m}{\bar{\hat z}^m} - \frac{\bar{\hat z}^m}{\hat z^m}\Big)
$$
$$
- 2(m_b(z^a z^a + \bar z^a\bar z^a) + m_a(z^b z^b + \bar z^b\bar z^b))\Big)
$$
$$
+ m_m^2\Big(\alpha+\beta-\tau-\frac{4}{3} - 2\Big(\alpha+\beta-\gamma-\delta-\frac{2}{9}\Big)\frac{\hat u^2}{\hat r^2}\Big)
$$
$$
+ (m_a^2 + m_b^2)\Big(\alpha-\beta+\tau-\frac{4}{3} - 2\Big(\alpha-\beta-\gamma+\delta-\frac{2}{9}\Big)\frac{\hat u^2}{\hat r^2}\Big)
$$
$$
- m_a m_b\Big(\alpha-\beta-\tau-1 - 2(\alpha-\beta-\gamma+\delta-1)\frac{\hat u^2}{\hat r^2}\Big)\Big(\frac{\hat z^m}{\bar{\hat z}^m} + \frac{\bar{\hat z}^m}{\hat z^m}\Big)\Big)
$$

(B.11)

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and

$$2(\hat{z}^m \partial_a \partial_{\bar{a}} \tilde{\bar{z}}^m - \hat{\bar{z}}^m \partial_a \partial_{\bar{a}} \tilde{z}^m)$$
$$= \frac{\zeta^2}{36} \hat{Z} \hat{u}^2 \Big(\frac{2}{\hat{r}^2} m_m \Big((1-\gamma) \Big((m_a(z^a z^a + \bar{z}^a \bar{z}^a) + m_b(z^b z^b + \bar{z}^b \bar{z}^b)) \Big(\frac{\hat{z}^m}{\hat{\bar{z}}^m} - \frac{\hat{\bar{z}}^m}{\hat{z}^m} \Big)$$
$$- (m_a(z^a z^a - \bar{z}^a \bar{z}^a) + m_b(z^b z^b - \bar{z}^b \bar{z}^b)) \Big(\frac{\hat{z}^m}{\hat{\bar{z}}^m} + \frac{\hat{\bar{z}}^m}{\hat{z}^m} \Big) \Big)$$
$$+ 2(\delta + \tau)(m_b(z^a z^a - \bar{z}^a \bar{z}^a) + m_a(z^b z^b - \bar{z}^b \bar{z}^b)) \Big)$$
$$- m_a m_b \Big(\alpha - \beta - \tau - 1 + 2(\gamma - \delta - \tau) \frac{\hat{u}^2}{\hat{r}^2} \Big) \Big(\frac{\hat{z}^m}{\hat{\bar{z}}^m} - \frac{\hat{\bar{z}}^m}{\hat{z}^m} \Big) \Big) \ .$$
$$(\text{B.12})$$

The linear combinations that appear in the above expressions on the l.h.s. are directly related to the radial and angular coordinate in a polar coordinate system. Expanding the embedding coordinates in the complex basis up to linear order in the corrections $\tilde{u}$ and $\tilde{\psi}$ as

$$\sqrt{2} z^m = u\, e^{i\psi} = (\hat{u} + \tilde{u})\, e^{i(\hat{\psi} + \tilde{\psi})} = (\hat{u} + \tilde{u} + i\hat{u}\tilde{\psi})\, e^{i\hat{\psi}} \ ,$$
$$\sqrt{2} \bar{z}^m = u\, e^{-i\psi} = (\hat{u} + \tilde{u})\, e^{-i(\hat{\psi} + \tilde{\psi})} = (\hat{u} + \tilde{u} - i\hat{u}\tilde{\psi})\, e^{-i\hat{\psi}} \ ,$$
$$(\text{B.13})$$

we find that the required linear combinations are given by

$$\hat{z}^m \partial_a \partial_{\bar{a}} \tilde{\bar{z}}^m + \hat{\bar{z}}^m \partial_a \partial_{\bar{a}} \tilde{z}^m = \hat{u} \partial_a \partial_{\bar{a}} \tilde{u} \ ,$$
$$\hat{z}^m \partial_a \partial_{\bar{a}} \tilde{\bar{z}}^m - \hat{\bar{z}}^m \partial_a \partial_{\bar{a}} \tilde{z}^m = -i\hat{u}^2 \partial_a \partial_{\bar{a}} \tilde{\psi} \ .$$
$$(\text{B.14})$$

Furthermore, with $\hat{u}^2 = 2\hat{z}^m \hat{\bar{z}}^m$, $\frac{\hat{z}^m}{\hat{\bar{z}}^m} = e^{2i\hat{\psi}}$ the combinations that appear on the r.h.s. of (B.11) and (B.12) can be expressed in terms of the angle $\hat{\psi}$ as

$$\frac{\hat{z}^m}{\hat{\bar{z}}^m} + \frac{\hat{\bar{z}}^m}{\hat{z}^m} = 2\cos 2\hat{\psi} \ , \qquad \frac{\hat{z}^m}{\hat{\bar{z}}^m} - \frac{\hat{\bar{z}}^m}{\hat{z}^m} = 2i\sin 2\hat{\psi} \ . \qquad (\text{B.15})$$

The combinations of the coordinates z^a and $\bar{z}^a$ are abbreviated in terms of four of the in total nine $l = 2$ $SO(4)$ spherical harmonics which are defined as

$$y_{++} = \frac{z^a z^a + z^b z^b + \bar{z}^a \bar{z}^a + \bar{z}^b \bar{z}^b}{\rho^2} = \frac{y^5 y^5 + y^6 y^6 - y^8 y^8 - y^9 y^9}{\rho^2} \ ,$$
$$y_{+-} = -i\frac{z^a z^a + z^b z^b - \bar{z}^a \bar{z}^a - \bar{z}^b \bar{z}^b}{\rho^2} = 2\frac{y^5 y^8 + y^6 y^9}{\rho^2} \ ,$$
$$y_{-+} = \frac{z^a z^a - z^b z^b + \bar{z}^a \bar{z}^a - \bar{z}^b \bar{z}^b}{\rho^2} = \frac{y^5 y^5 - y^6 y^6 - y^8 y^8 + y^9 y^9}{\rho^2} \ ,$$
$$y_{--} = -i\frac{z^a z^a - z^b z^b - \bar{z}^a \bar{z}^a + \bar{z}^b \bar{z}^b}{\rho^2} = 2\frac{y^5 y^8 - y^6 y^9}{\rho^2} \ ,$$
$$(\text{B.16})$$

where in the second equalities we have fixed the indices to $a = 2$, $b = 3$ and $m = 1$. This corresponds to an embedding of the D7-brane as shown in table 1. Using the following parameterization of the real coordinates

$$y^5 = \rho \cos \theta \cos \phi_1 \ , \quad y^6 = \rho \cos \theta \sin \phi_1 \ , \quad y^8 = \rho \sin \theta \cos \phi_2 \ , \quad y^9 = \rho \sin \theta \sin \phi_2 \ ,$$

$$\tag{B.17}$$

where $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi_{1,2} \leq 2\pi$, the spherical harmonics become

$$
\begin{aligned}
y_{++} &= \cos 2\theta \ , \\
y_{+-} &= \sin 2\theta \cos(\phi_1 - \phi_2) \ , \\
y_{-+} &= \cos^2 \theta \cos 2\phi_1 - \sin^2 \theta \cos 2\phi_2 \ , \\
y_{--} &= \sin 2\theta \cos(\phi_1 + \phi_2) \ .
\end{aligned}
\tag{B.18}
$$

The equations of motion then read

$$
\begin{aligned}
2\partial_a \partial_{\bar{a}} \tilde{u} = \frac{\zeta^2}{36} \hat{Z} \hat{u} \Big(& 2\frac{\rho^2}{\hat{r}^2}(1 - \gamma) m_1 \big(-(m_2 + m_3) y_{++}(1 - \cos 2\hat{\psi}) \\
& + (m_2 - m_3) y_{-+}(1 + \cos 2\hat{\psi}) \\
& + ((m_2 + m_3) y_{+-} + (m_2 - m_3) y_{--}) \sin 2\hat{\psi}\big) \\
& + m_1^2 \Big(\alpha + \beta - \tau - \frac{4}{3} - 2\Big(\alpha + \beta - \gamma - \delta - \frac{2}{9}\Big) \frac{\hat{u}^2}{\hat{r}^2} \Big) \\
& + (m_2^2 + m_3^2)\Big(\alpha - \beta + \tau - \frac{4}{3} - 2\Big(\alpha - \beta - \gamma + \delta - \frac{2}{9}\Big)\frac{\hat{u}^2}{\hat{r}^2}\Big) \\
& - 2m_2 m_3 \Big(\alpha - \beta - \tau - 1 - 2(\alpha - \beta - \gamma + \delta - 1)\frac{\hat{u}^2}{\hat{r}^2}\Big) \cos 2\hat{\psi} \Big)
\end{aligned}
$$

$$
\begin{aligned}
2\partial_a \partial_{\bar{a}} \tilde{\psi} = \frac{\zeta^2}{36} \hat{Z} \Big(& 2\frac{\rho^2}{\hat{r}^2} m_1 \big((\gamma - 1)((m_2 + m_3) y_{++} + (m_2 - m_3) y_{-+}) \sin 2\hat{\psi} \\
& - (m_2 + m_3) y_{+-}(\delta + \tau + (\gamma - 1) \cos 2\hat{\psi}) \\
& + (m_2 - m_3) y_{--}(\delta + \tau - (\gamma - 1) \cos 2\hat{\psi})\big) \\
& + 2m_2 m_3 \Big(\alpha - \beta - \tau - 1 + 2(\gamma - \delta - \tau)\frac{\hat{u}^2}{\hat{r}^2}\Big) \sin 2\hat{\psi} \Big) \ .
\end{aligned}
$$

$$\tag{B.19}$$

For both, u and ψ they have the same structure which is compactly summarized as

$$2\partial_a\partial_{\bar{a}}f = \frac{n_f}{\hat{r}^4}\left(B_f + C_f\frac{\hat{u}^2}{\hat{r}^2} - (C_f^{++}y_{++} + C_f^{+-}y_{+-} + C_f^{-+}y_{-+} + C_f^{--}y_{--})\frac{\rho^2}{\hat{r}^2}\right)\,,$$

(B.20)

where $f = u$ or $f = \psi$, $n_u = \hat{u}$ or respectively $n_\psi = 1$, and the constants are given by

$$B_u = \frac{\zeta^2 R^4}{216}\big((m_2 + m_3)^2(6\tau - 1) + (m_2 - m_3)^2(6(\alpha - \beta) - 7)$$

$$+ 2m_1^2(3(\alpha + \beta - \tau) - 4) + 12m_2 m_3(\alpha - \beta - \tau - 1)(1 - \cos 2\hat{\psi}))\,,$$

$$C_u = -\frac{\zeta^2 R^4}{108}\left((m_2 + m_3)^2\frac{7}{3} + (m_2 - m_3)^2\left(6(\alpha - \beta - \gamma + \delta) - \frac{11}{3}\right)\right.$$

$$+ 2m_1^2\left(3(\alpha + \beta - \gamma - \delta) - \frac{2}{3}\right)$$

$$\left. + 12m_2 m_3(\alpha - \beta - \gamma + \delta - 1)(1 - \cos 2\hat{\psi})\right)\,,$$

$$C_u^{++} = -\frac{\zeta^2 R^4}{18}m_1(m_2 + m_3)(\gamma - 1)(1 - \cos 2\hat{\psi})\,,$$

$$C_u^{+-} = \frac{\zeta^2 R^4}{18}m_1(m_2 + m_3)(\gamma - 1)\sin 2\hat{\psi}\,,$$

$$C_u^{-+} = \frac{\zeta^2 R^4}{18}m_1(m_2 - m_3)(\gamma - 1)(1 + \cos 2\hat{\psi})\,,$$

$$C_u^{--} = \frac{\zeta^2 R^4}{18}m_1(m_2 - m_3)(\gamma - 1)\sin 2\hat{\psi}\,,$$

(B.21)

or respectively by

$$B_\psi = \frac{\zeta^2 R^4}{18}m_2 m_3(\alpha - \beta - \tau - 1)\sin 2\hat{\psi}\,,$$

$$C_\psi = \frac{\zeta^2 R^4}{9}m_2 m_3(\gamma - \delta - \tau)\sin 2\hat{\psi}\,,$$

$$C_\psi^{++} = -\frac{\zeta^2 R^4}{18}m_1(m_2 + m_3)(\gamma - 1)\sin 2\hat{\psi}\,,$$

$$C_\psi^{+-} = \frac{\zeta^2 R^4}{18}m_1(m_2 + m_3)(\gamma + \delta + \tau - 1 - (\gamma - 1)(1 - \cos 2\hat{\psi}))\,,$$

$$C_\psi^{-+} = -\frac{\zeta^2 R^4}{18}m_1(m_2 - m_3)(\gamma - 1)\sin 2\hat{\psi}$$

$$C_\psi^{--} = \frac{\zeta^2 R^4}{18}m_1(m_2 - m_3)(\gamma - \delta - \tau - 1 - (\gamma - 1)(1 - \cos 2\hat{\psi}))\,.$$

(B.22)

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C   Explicit expansion of the action

In this section we derive the explicit expression for the action (5.1). We need the corrections to the dilaton (2.15) in complex coordinates, as well as the combination of the subtraces (A.10) of the corrections to the metric. Furthermore, we need the form combination (A.4). The components of the correction to the metric (A.11) and of the form combination (A.5) have to be contracted with the derivatives of the embedding coordinates. Finally, (2.13) serves to replace the parameters in the metric by their explicit values.

The combination of non-derivative terms is then found to be given by

$$
\begin{aligned}
\tilde{\phi} &+ \frac{1}{2} Z^{-\frac{1}{2}} (Z \tilde{g}_{\mu\mu} + \tilde{g}_{aa}) + \frac{1}{2} \mathrm{e}^{-\hat{\phi}} Z^{-1} (\alpha - \beta \star_4) B \cdot B + \frac{\tau}{2} \mathrm{e}^{\hat{\phi}} Z^{-1} \star_4 \tilde{C}_2 \cdot \tilde{C}_2 \\
&= \frac{\zeta^2}{36} Z \Big(- (\alpha - \beta - \tau - 1) m_a m_b (z^m z^m + \bar{z}^m \bar{z}^m) \\
&\quad + m_a m_m (z^b z^b + \bar{z}^b \bar{z}^b) + m_b m_m (z^a z^a + \bar{z}^a \bar{z}^a) + \frac{5}{18} M^2 \rho^2 \\
&\quad + \frac{1}{18} ((9(\alpha + \beta - \tau) - 2) m_m^2 + (9(\alpha - \beta + \tau) - 2)(m_a^2 + m_b^2)) u^2 \Big) \ .
\end{aligned}
\tag{C.1}
$$

While on the l.h.s. the indices μ, a are summed over, on the r.h.s. a, b, m take fixed distinct values. After some manipulations thereby using also (B.10) to obtain simple linear combinations of the derivative terms, the remaining contributions to the action

combine as

$$Z^{-\frac{1}{2}}\partial_a \tilde{y}^m \tilde{g}_{am} + 2\,\mathrm{e}^{-\hat{\phi}}\,Z^{-1}\big((\gamma - \delta\star_4)B\cdot\partial\tilde{y}B + \tau\,\mathrm{e}^{2\hat{\phi}}\star_4\tilde{C}_2\cdot\partial\tilde{y}\tilde{C}_2\big)$$

$$= -\frac{\zeta^2}{72}Z\Big(\Big(\Big(\gamma - \delta + \tau - \frac{5}{9}\Big)(m_a^2 + m_b^2) + \Big(\gamma + \delta - \tau - \frac{5}{9}\Big)m_m^2\Big)$$

$$(z^a\partial_a + \bar{z}^a\partial_{\bar{a}})(\hat{z}^m\tilde{\bar{z}}^m + \hat{\bar{z}}^m\tilde{z}^m)$$

$$+ \big((\gamma - \delta + \tau - 1)(m_a^2 - m_b^2) + (\gamma + \delta - \tau - 1)m_m^2\big)$$

$$(z^a\partial_a - \bar{z}^a\partial_{\bar{a}})(\hat{z}^m\tilde{\bar{z}}^m - \hat{\bar{z}}^m\tilde{z}^m) \tag{C.2}$$

$$- 2(\gamma - \delta - \tau)m_a m_b(z^a\partial_a + \bar{z}^a\partial_{\bar{a}})(\hat{z}^m\tilde{z}^m + \hat{\bar{z}}^m\tilde{\bar{z}}^m)$$

$$- 2m_m m_b(\gamma(z^a\partial_{\bar{a}} + \bar{z}^a\partial_a)(\hat{z}^m\tilde{\bar{z}}^m + \hat{\bar{z}}^m\tilde{z}^m)$$

$$- (\delta + \tau)(z^a\partial_{\bar{a}} - \bar{z}^a\partial_a)(\hat{z}^m\tilde{\bar{z}}^m - \hat{\bar{z}}^m\tilde{z}^m))$$

$$+ 4(\gamma - 1)m_a m_m(z^a\partial_{\bar{a}}(\hat{\bar{z}}^m\tilde{\bar{z}}^m) + \bar{z}^a\partial_a(\hat{z}^m\tilde{z}^m))\Big)\,,$$

where on the l.h.s. for compactness we have used real coordinates and all indices are summed over independently, while on the r.h.s. which is expressed in complex coordinates the summation runs over a and b, such that a, b and m take distinct values.

To obtain the action as an expansion in terms of the corrections which appear in the decomposition (4.5), we have to expand the warp factor up to linear order in $\tilde{z}^m$ as

$$Z = \hat{Z}\Big(1 - \frac{4}{\hat{r}^2}(\hat{z}^m\tilde{\bar{z}}^m + \hat{\bar{z}}^m\tilde{z}^m)\Big)\,, \qquad \hat{Z} = \frac{R^4}{\hat{r}^4}\,, \tag{C.3}$$

where a 'hat' indicates that the corresponding expression has to be evaluated with the unperturbed value $\hat{z}^m$. Introducing polar coordinates as in (B.13) and embedding the

D7-brane as in table 1, the Lagrangian is then given by

$$-\frac{\mathrm{e}^{\hat{\phi}}}{T_7}\mathcal{L} = 1 + \partial_a \tilde{u}\partial_{\bar{a}}\tilde{u} + \hat{u}^2\partial_a\tilde{\psi}\partial_{\bar{a}}\tilde{\psi}$$

$$+ \frac{\zeta^2}{72}\hat{Z}\Big(\big((m_2-m_3)^2(\alpha-\beta-1) + (m_2+m_3)^2\tau\big)\hat{u}\big(\hat{u} + 2\big(1 - 2\tfrac{\hat{u}^2}{\hat{r}^2}\big)\tilde{u}\big)$$

$$- \big((m_2-m_3)^2(\gamma-\delta) + (m_2+m_3)^2\tau\big)(z^a\partial_a + \bar{z}^a\partial_{\bar{a}})\hat{u}\tilde{u}$$

$$+ \frac{1}{3}(m_2^2+m_3^2)\Big(\tfrac{2}{3}\hat{u}^2 + \tfrac{5}{3}\hat{r}^2 - 2\big(1 + \tfrac{4}{3}\tfrac{\hat{u}^2}{\hat{r}^2}\big)\hat{u}\tilde{u} + \tfrac{5}{3}(z^a\partial_a + \bar{z}^a\partial_{\bar{a}})\hat{u}\tilde{u}\Big)$$

$$+ m_1^2\Big(\big(\alpha+\beta-\tau-\tfrac{7}{9}\big)\hat{u}^2 + \tfrac{5}{9}\hat{r}^2 + 2\big(\alpha+\beta-\tau-\tfrac{4}{3} - 2\big(\alpha+\beta-\tau-\tfrac{7}{9}\big)\tfrac{\hat{u}^2}{\hat{r}^2}\big)\hat{u}\tilde{u}$$

$$- \big(\gamma+\delta-\tau-\tfrac{5}{9}\big)(z^a\partial_a + \bar{z}^a\partial_{\bar{a}})\hat{u}\tilde{u}\Big)$$

$$+ i\big((m_2^2-m_3^2)(\gamma-\delta+\tau-1)(z^2\partial_2 - \bar{z}^2\partial_{\bar{2}} - z^3\partial_3 + \bar{z}^3\partial_{\bar{3}})$$

$$+ m_1^2(\gamma+\delta-\tau-1)(z^a\partial_a - \bar{z}^a\partial_{\bar{a}})\big)\hat{u}^2\tilde{\psi}$$

$$+ 2m_2m_3\Big((\alpha-\beta-\tau-1)\hat{u}\big(\hat{u}(1-\cos 2\hat{\psi}) + 2\big(1 - 2\tfrac{\hat{u}^2}{\hat{r}^2}\big)(1-\cos 2\hat{\psi})\tilde{u} + 2\hat{u}\sin 2\hat{\psi}\,\tilde{\psi}\big)$$

$$- (\gamma-\delta-\tau)(z^a\partial_a + \bar{z}^a\partial_{\bar{a}})\hat{u}\big((1-\cos 2\hat{\psi})\tilde{u} + \hat{u}\sin 2\hat{\psi}\,\tilde{\psi}\big)\Big)$$

$$+ m_1(m_2+m_3)\Big((\hat{r}^2-\hat{u}^2)y_{++}\big(1 - \tfrac{4}{\hat{r}^2}\hat{u}\tilde{u}\big)$$

$$+ \hat{u}(z^a\partial_{\bar{a}} + \bar{z}^a\partial_a)(\tilde{u} + (\gamma-1)((1-\cos 2\hat{\psi})\tilde{u} + \hat{u}\sin 2\hat{\psi}\,\tilde{\psi}))$$

$$+ i\hat{u}(z^a\partial_{\bar{a}} - \bar{z}^a\partial_a)((\gamma+\delta+\tau-1)\hat{u}\tilde{\psi} + (\gamma-1)(\sin 2\hat{\psi}\,\tilde{u} - (1-\cos 2\hat{\psi})\hat{u}\tilde{\psi}))\Big)$$

$$- m_1(m_2-m_3)\Big((\hat{r}^2-\hat{u}^2)y_{-+}\big(1 - \tfrac{4}{\hat{r}^2}\hat{u}\tilde{u}\big)$$

$$+ \hat{u}(z^2\partial_{\bar{2}} + \bar{z}^2\partial_2 - z^3\partial_3 - \bar{z}^3\partial_3)((2\gamma-1)\tilde{u} - (\gamma-1)((1-\cos 2\hat{\psi})\tilde{u} + \hat{u}\sin 2\hat{\psi}\,\tilde{\psi}))$$

$$- i\hat{u}(z^2\partial_{\bar{2}} - \bar{z}^2\partial_2 - z^3\partial_{\bar{3}} + \bar{z}^3\partial_3)((\gamma-\delta-\tau-1)\hat{u}\tilde{\psi}$$

$$+ (\gamma-1)(\sin 2\hat{\psi}\,\tilde{u} - (1-\cos 2\hat{\psi})\hat{u}\tilde{\psi}))\Big)\,,$$

$$\tag{C.4}$$

where a summation over $a = 2,3$ on both sides is understood. Finally, we partially integrate all terms which contain derivatives of $\tilde{u}$ and $\tilde{\psi}$. We require the expressions

$$(\partial_a z^a + \partial_{\bar{a}}\bar{z}^a)\hat{Z} = 4\hat{Z}\frac{\hat{u}^2}{\hat{r}^2}\,, \qquad (\partial_{\bar{a}}z^a + \partial_a\bar{z}^a)\hat{Z} = -4\hat{Z}\frac{\rho^2}{\hat{r}^2}y_{++}\,,$$

$$-\frac{i}{2}(\partial_a z^a - \partial_{\bar{a}}\bar{z}^a)\hat{Z} = 0\,, \qquad -i(\partial_{\bar{a}}z^a - \partial_a\bar{z}^a)\hat{Z} = -4\hat{Z}\frac{\rho^2}{\hat{r}^2}y_{+-}\,,$$

$$\tag{C.5}$$

and

$$(\partial_{\bar{2}}z^2 - \partial_{\bar{3}}z^3 + \partial_2 \bar{z}^2 - \partial_3 \bar{z}^3)\hat{Z} = -4\hat{Z}\frac{\rho^2}{\hat{r}^2}y_{-+} \ ,$$

$$-i(\partial_{\bar{2}}z^2 - \partial_{\bar{3}}z^3 - \partial_2 \bar{z}^2 + \partial_3 \bar{z}^3)\hat{Z} = -4\hat{Z}\frac{\rho^2}{\hat{r}^2}y_{--} \ ,$$

(C.6)

where the derivatives act on all functions on the right, and the $SO(4)$ spherical harmonics that appear on the r.h.s. are defined in (B.16). The Lagrangian can then be cast into the compact form

$$
\begin{aligned}
-\frac{\mathrm{e}^{\hat{\phi}}}{T_7}\mathcal{L} = {}& 1 + \partial_a\tilde{u}\partial_{\bar{a}}\tilde{u} + \hat{u}^2\partial_a\tilde{\psi}\partial_{\bar{a}}\tilde{\psi} + \frac{B_u}{2}\frac{\hat{u}^2}{\hat{r}^4} \\
& + \frac{\zeta^2}{216}\hat{Z}\Big(\frac{5}{3}M^2(\hat{r}^2 + \hat{u}^2) + 3m_1((m_2 + m_3)y_{++} - (m_2 - m_3)y_{-+})\rho^2\Big) \\
& + \Big(B_u + C_u\frac{\hat{u}^2}{\hat{r}^2} - (C_u^{++}y_{++} + C_u^{+-}y_{+-} + C_u^{-+}y_{-+} + C_u^{--}y_{--})\frac{\rho^2}{\hat{r}^2}\Big)\frac{\hat{u}}{\hat{r}^4}\tilde{u} \\
& + \Big(B_\psi + C_\psi\frac{\hat{u}^2}{\hat{r}^2} - (C_\psi^{++}y_{++} + C_\psi^{+-}y_{+-} + C_\psi^{-+}y_{-+} + C_\psi^{--}y_{--})\frac{\rho^2}{\hat{r}^2}\Big)\frac{\hat{u}^2}{\hat{r}^4}\tilde{\psi} \\
& + \text{total derivatives} \ ,
\end{aligned}
$$

(C.7)

where M^2 can be found in (2.12), and the constants are defined in (B.21) and (B.22). It is easy to verify that the equations of motion derived from this Lagrangian coincide with the ones given in (5.8) and (B.19).

D   Analytic solution of the equations of motion

The equations of motion (5.8) and (B.19) have the structure

$$2\partial_a\partial_{\bar{a}}f = \frac{n_f}{\hat{r}^4}\Big(B_f + C_f\frac{\hat{u}^2}{\hat{r}^2} - C_f^I y_I\frac{\rho^2}{\hat{r}^2}\Big) = \frac{n_f}{\hat{r}^4}\Big(B_f + C_f - (C_f + C_f^I y_I)\frac{\rho^2}{\hat{r}^2}\Big) \ , \qquad \text{(D.1)}$$

where $n_u = \hat{u}$, $n_\psi = 1$, and we sum over I which distinguishes the level $l = 2$ $SO(4)$ spherical harmonics defined in (B.16).

To find a solution of the above equation, we recall the action of the Laplace operator in d-dimensional flat space when it acts on a function $f(\rho, \theta_i) = f^I(\rho)Y_I(\theta_i)$, $i = 1, \ldots, d-1$, where ρ is the radial coordinate and θ_i denote the angle coordinates. The spherical harmonics Y_I (representations of $SO(d)$) carry labels $I = (l, m_1 \ldots m_{d-2})$, including also the case $l = 0$. The Laplace operator acts as

$$\partial_a\partial_a f = \sum_I \Big(\frac{1}{\rho^{d-1}}\partial_\rho(\rho^{d-1}\partial_\rho) - \frac{l(l + d - 2)}{\rho^2}\Big)f^I Y_I \ , \qquad \text{(D.2)}$$

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where on the l.h.s. we sum over $a = 1, \ldots, d$. We separate from the radial dependent coefficients of the level l spherical harmonics a factor ρ^l by rewriting $f^I = \rho^l h_f^I$. This yields the relation

$$\left(\frac{1}{\rho^{d-1}}\partial_\rho(\rho^{d-1}\partial_\rho) - \frac{l(l+d-2)}{\rho^2}\right)f^I = \frac{1}{\rho^{l+d-1}}\partial_\rho\left(\rho^{2l+d-1}\partial_\rho h_f^I\right) , \qquad (D.3)$$

i.e. rewritten in terms of the functions h_f^I, the Laplace operator only generates first and second derivative of h_f^I in a nested manner, but does not leave the h_f^I without derivatives.

Using this transformation in the special case of $d = 4$, the equations of motion (D.1) are rewritten as

$$\frac{1}{\rho^{l+3}}\partial_\rho\left(\rho^{2l+3}\partial_\rho h_f^I\right) = \frac{n_f}{\hat{r}^4}\left(B_f^I + C_f^I - C_f^I\frac{\rho^2}{\hat{r}^2}\right) . \qquad (D.4)$$

We have split it into individual equations for each value of I. For the coefficients $h_f(\rho)$ of the constant spherical harmonic ($l = 0$) and for $h_f^{l=2}(\rho)$ for the level $l = 2$ spherical harmonics we respectively find

$$\begin{aligned}
\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho h_f) &= \frac{n_f}{\hat{r}^4}\left(B_f + C_f - C_f\frac{\rho^2}{\hat{r}^2}\right) , \\
\frac{1}{\rho^5}\partial_\rho(\rho^7\partial_\rho h_f^{l=2}) &= -n_f C_f^{l=2}\frac{\rho^2}{\hat{r}^6} .
\end{aligned} \qquad (D.5)$$

Performing the first step of integration, the results read

$$\begin{aligned}
\rho^3\partial_\rho h_f &= \frac{n_f}{2}\left((B_f - C_f)\frac{\hat{u}^2}{\hat{r}^2} + B_f\ln\hat{r}^2 + C_f\frac{\hat{u}^4}{2\hat{r}^4} - 2A_f\right) , \\
\rho^7\partial_\rho h_f^{l=2} &= -\frac{n_f}{2}\left(C_f^{l=2}\left(\rho^2 - 3\frac{\hat{u}^4}{\hat{r}^2} + \frac{\hat{u}^6}{2\hat{r}^4} - 3\hat{u}^2\ln\hat{r}^2\right) + 6A_f^{l=2}\hat{u}^2\right) ,
\end{aligned} \qquad (D.6)$$

where A_f and $A_f^{l=2}$ denote integration constants. After the second integration we obtain

$$\begin{aligned}
h_f &= \hat{h}_f - \frac{n_f}{2}\left((2B_f - C_f)\frac{1}{4\rho^2} + B_f\frac{1}{2\rho^2}\ln\hat{r}^2 + C_f\frac{1}{4\hat{r}^2} - A_f\frac{1}{\rho^2}\right) , \\
h_f^{l=2} &= \hat{h}_f^{l=2} + \frac{n_f}{2\rho^2}\left(C_f^{l=2}\left(\frac{1}{2\rho^2}\left(1 - \frac{\hat{u}^2}{\rho^2}\ln\hat{r}^2\right) - \frac{5}{12}\frac{\hat{u}^2}{\rho^4} - \frac{1}{4\hat{r}^2}\right) + A_f^{l=2}\frac{\hat{u}^2}{\rho^4}\right) ,
\end{aligned} \qquad (D.7)$$

where $\hat{h}_f$ and $\hat{h}_f^{l=2}$ are the corresponding integration constants. In general the above given functions diverge in the limit $\rho \to 0$. However, for appropriately chosen constants

$$A_f = \frac{1}{2}B_f(1 + \ln\hat{u}^2) - \frac{1}{4}C_f , \qquad A_f^{l=2} = \frac{1}{12}C_f^{l=2}(5 + 6\ln\hat{u}^2) , \qquad (D.8)$$

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the functions become regular at $\rho = 0$, i.e. the D7-brane embeddings are regular at $\rho = 0$ in this case. The coefficient functions assume the form

$$
\begin{aligned}
h_f &= \hat{h}_f - \frac{n_f}{8}\Big(B_f\frac{2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + C_f\frac{1}{\hat{r}^2}\Big)\,, \\
h_f^{l=2} &= \hat{h}_f^{l=2} + \frac{n_f}{8\rho^2}C_f^{l=2}\Big(\frac{2}{\rho^2}\Big(1 - \frac{\hat{u}^2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2}\Big) - \frac{1}{\hat{r}^2}\Big)\,.
\end{aligned}
\tag{D.9}
$$

In the full solution $h_f^{l=2}$ is multiplied by ρ^2. For $\rho \to \infty$ it is therefore only regular if $\hat{h}_f^{l=2} = 0$. The everywhere regular solution of (D.1) hence depends on a single integration constant $\hat{f} = \hat{h}_f$. Its final form and its first and second derivative are given by

$$
\begin{aligned}
f &= \hat{f} - \frac{n_f}{8}\Big(B_f\frac{2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + C_f\frac{1}{\hat{r}^2} - C_f^I\Big(\frac{2}{\rho^2}\Big(1 - \frac{\hat{u}^2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2}\Big) - \frac{1}{\hat{r}^2}\Big)y_I\Big) \\
\partial_\rho f &= \frac{n_f}{4}\Big(B_f\frac{2}{\rho}\Big(\frac{1}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} - \frac{1}{\hat{r}^2}\Big) + C_f\frac{\rho}{\hat{r}^4} - C_f^I\Big(\frac{2}{\rho^3}\Big(1 - 2\frac{\hat{u}^2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + \frac{\hat{u}^2}{\hat{r}^2}\Big) - \frac{\rho}{\hat{r}^4}\Big)y_I\Big) \\
\partial_\rho^2 f &= \frac{n_f}{4}\Big(B_f\frac{2}{\rho^2}\Big(-\frac{3}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2} + \frac{3}{\hat{r}^2} + 2\frac{\rho^2}{\hat{r}^4}\Big) + C_f\frac{1}{\hat{r}^4}\Big(1 - 4\frac{\rho^2}{r^2}\Big) \\
&\qquad - C_f^I\Big(-\frac{20}{\rho^4}\Big(1 - \frac{\hat{u}^2}{\rho^2}\ln\frac{\hat{r}^2}{\hat{u}^2}\Big) + \frac{1}{\rho^2\hat{r}^2}\Big(10 + 3\frac{\rho^2}{\hat{r}^2}\Big) + 4\frac{\rho^2}{\hat{r}^6}\Big)y_I\Big)\,.
\end{aligned}
\tag{D.10}
$$

For a closer analysis it is necessary to understand the asymptotic behaviour of the above expressions. In the limits $\rho \ll \hat{u}$ and $\rho \gg \hat{u}$ we find

$$
\begin{aligned}
f &= \begin{cases} \hat{f} - \frac{n_f}{8\hat{u}^2}(2B_f + C_f) & \rho \to 0 \\ \hat{f} - \frac{n_f}{8\rho^2}\big(2B_f\ln\frac{\rho^2}{\hat{u}^2} + C_f - C_f^I y_I\big) & \rho \to \infty \end{cases}, \\
\partial_\rho f &= \begin{cases} 0 & \rho \to 0 \\ \frac{n_f}{4\rho^3}\big(2B_f(\ln\frac{\rho^2}{\hat{u}^2} - 1) + C_f - C_f^I y_I\big) & \rho \to \infty \end{cases}, \\
\partial_\rho^2 f &= \begin{cases} \frac{n_f}{12\hat{u}^2}(3(B_f + C_f) + C_f^I y_I) & \rho \to 0 \\ \frac{n_f}{4\rho^4}(2B_f(-3\ln\frac{\rho^2}{\hat{u}^2} + 5) - 3C_f + 3C_f^I y_I) & \rho \to \infty \end{cases}.
\end{aligned}
\tag{D.11}
$$

In particular, $\hat{f}$ is the constant value of f at the boundary at $\rho \to \infty$.

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